Walking the Tightrope between Expressiveness and Uncomputability: AGM Contraction beyond the Finitary Realm (Extended Version)

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Abstract
Although there has been significant interest in extending the AGM paradigm of belief change beyond finitary logics, the computational aspects of AGM have remained almost untouched. We investigate the computability of AGM contraction on non-finitary logics, and show an intriguing negative result: there are infinitely many uncomputable AGM contraction functions in such logics. Drastically, even if we restrict the theories used to represent epistemic states, in all non-trivial cases, the uncomputability remains. On the positive side, we use Büchi automata to construct computable AGM contraction functions on Linear Temporal Logic (LTL).

1. Introduction

The field of *Belief Change* [1, 2, 3] investigates how to keep a corpus of beliefs consistent as it evolves. The field is mainly founded on the AGM paradigm [1], named after its authors' initials, which distinguishes, among others, two main kinds of changes: *belief revision*, which consists in incorporating an incoming piece of information with the proviso that the updated corpus of beliefs is consistent; and *belief contraction* whose purpose is to retract an obsolete piece of information. In either case, the incurred changes should be minimized so that most of the original beliefs are preserved. This is known as *principle of minimal change*. Contraction is central as it can be used to define other forms of belief change. For example, belief revision can be defined in terms of contraction: first, remove information in conflict with the incoming belief via contraction, only then incorporate the incoming belief. When classical negation is at disposal, this recipe for defining revision from contraction is formalised via the Levi identity [4, 5].

The AGM paradigm prescribes rationality postulates that capture the *principle of minimal change*, and constructive functions that satisfy such postulates, called rational functions, as for instance *partial meet* [1], (smooth) *kernel contraction* [6], epistemic entrenchment [2] and Grove's system of spheres [7]. Originally, the AGM paradigm was developed assuming some conditions about the underlying logic [2, 8]. Although these conditions cover some classical logics such as classical Propositional Logic and First Order Logic, they restrict the reach of the AGM paradigm into more expressive logics including several Descriptions Logics [9], Modal Logics [10] and Temporal Logics such as LTL, CTL and CTL* [11]. It turns out that the AGM paradigm is independent of such conditions [8], although rational contraction functions do not exist in every logic [8, 12]. Logics in which rational contraction functions do exist are dubbed *AGM compliant* [8]. As a result, several works have been dedicated to dispense with the AGM assumptions in order to extend the paradigm to more expressive logics: Horn logics [13, 14, 15], para-consistent logics [16], Description Logics [17, 12, 8], and non-compact logics [18]. See [19] for a discussion of several other works in this line.

Although much effort has been put into extending the AGM paradigm to more expressive logics, few works have investigated the computational aspects of the AGM paradigm such as [20, 21, 22]. All these works, however, focused on investigating the complexity of decision problems for some fixed belief change operators on classical propositional logics and Horn. In light of the interest and effort of expanding the AGM paradigm for more expressive non-classical logics, it is paramount to comprehend the computational aspects of belief change in such more expressive logics. In this context, there is a central question that even precedes complexity:

Computability / Effectiveness: Given a belief change operator $\circ$, does there exist a Turing Machine that computes $\circ$, and stops on all inputs?

The answer to the question above depends on two main elements: the underlying logic, and the chosen operator. Clearly, it only makes sense to answer such questions for logics that are AGM compliant. However, independently of the operator, the question is trivial for finitary logics, that is, logics whose language contains only finitely many equivalence classes, as it is the case of classical propositional logics. For non-finitary logics, by contrast, we show a disruptive result: AGM rational contraction functions suffer from uncomputability.

This first result uses all the expressive power of the underlying logic. To control computability, one could limit the space of epistemic states to some specific set of theories (logically closed set of sentences). However, we show that no matter how much we constrain the space of epistemic states, uncomputability still remains, as long as the restriction is not so severe that the space collapses back to the finitary case. Although this shows that uncomputability is unavoidable in all such expressive spaces, it is of extreme importance to identify how, and under which conditions, one can construct specific (families of) contraction functions that are computable. We investigate this question for Linear Temporal Logic (LTL), and we show that when representing epistemic states via *Büchi automata* [23], we can construct families of contraction functions that are computable within such a space. LTL is a very expressive logic used in a plethora of applications in Computer Science and AI. For example, LTL has been used for specification and

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verification of software and hardware systems [11], in business process models such as DECLARE [24], in planning and reasoning about actions [25, 26], and extending Description Logics with temporal knowledge [27, 28]. Büchi automata are endued with closure properties which allow for both effective reasoning and computable contraction functions.

Roadmap: In Section 2, we review basic concepts regarding logics, including LTL and Büchi automata. We briefly review AGM contraction in Section 3. Section 4 discusses the question of finite representation for epistemic states, and presents our first contribution, namely, we introduce a general notion to capture all forms of finite representations, and show a negative result: for a wide class of so-called *compendious* logics, not all epistemic states can be represented finitely. In Section 5, we present an expressive method of finite representation for LTL which is based on Büchi automata, and discuss how it supports reasoning. Section 6 introduces the notion of *AGM closedness*, i.e., every rational contraction outcome on a finitely representable belief state should again be finitely representable. We show that, under certain weak conditions, closedness cannot be satisfied for compendious logics. In Section 7, we establish our third negative result for compendious logics: even if we restrict ourselves to contraction functions whose output can be represented, uncomputability of contraction is inevitable in the non-finitary case, i.e., there always exist uncountably many uncomputable contraction functions. On the positive side, in Section 8, we show that computable contractions do exist for LTL theories represented via Büchi automata, and we identify the conditions needed for computability. Section 9 discusses the impact of our results and provides an outlook on future work.

2. Logics and Automata

We review a general notion of logics that will be used throughout the paper. We use $\mathcal{P}(X)$ to denote the power set of a set X. A *logic* is a pair $\mathbb{L} = (Fm, Cn)$ comprising a countable[1] set of *formulae* Fm, and a *consequence operator* $Cn : \mathcal{P}(Fm) \to \mathcal{P}(Fm)$ that maps each set of formulae to the conclusions entailed from it. We sometimes write $Fm_{\mathbb{L}}$ and $Cn_{\mathbb{L}}$ for brevity.

We consider logics that are *Tarskian*, that is, logics whose consequence operator Cn is monotone (if $X_1 \subseteq X_2$ then $Cn(X_1) \subseteq Cn(X_2)$), extensive ($X \subseteq Cn(X)$) and idempotent ($Cn(Cn(X)) = Cn(X)$). We say that two formulae $\varphi, \psi \in Fm$ are logically equivalent, denoted $\varphi \equiv \psi$, if $Cn(\varphi) = Cn(\psi)$. $Cn(\emptyset)$ is the set of all tautologies. A *theory* of $\mathbb{L}$ is a set of formulae $\mathcal{K}$ such that $Cn(\mathcal{K}) = \mathcal{K}$. The expansion of a theory $\mathcal{K}$ by a formula φ is the theory $\mathcal{K} + \varphi := Cn(\mathcal{K} \cup \{\varphi\})$. Let $\mathsf{Th}_{\mathbb{L}}$ denote the set of all theories of $\mathbb{L}$. If $\mathsf{Th}_{\mathbb{L}}$ is finite, we say that $\mathbb{L}$ is *finitary*; otherwise, $\mathbb{L}$ is *non-finitary*. Equivalently, $\mathbb{L}$ is finitary if $\mathbb{L}$ has only finitely many formulae up to logical equivalence.

A theory $\mathcal{K}$ is *consistent* if $\mathcal{K} \neq Fm$, and it is *complete* if for all formulae $\varphi \notin \mathcal{K}$, we have $\mathcal{K} + \varphi = Fm$. The set of all complete consistent theories of $\mathbb{L}$ is denoted as $CCT_{\mathbb{L}}$. The set of all CCTs that do not contain φ is given by $\overline{\omega}(\varphi)$.

A logic $\mathbb{L}$ is *Boolean*, if $Fm_{\mathbb{L}}$ is closed under the classical boolean operators and they are interpreted as usual. In particular, for a logic to be Boolean, we require every theory $\mathcal{K} \in \mathsf{Th}_{\mathbb{L}}$ to coincide with the intersection of all the CCTs containing $\mathcal{K}$, that is, $\mathcal{K} = \bigcap\{\, \mathcal{K}' \in CCT_{\mathbb{L}} \mid \mathcal{K} \subseteq \mathcal{K}' \,\}$.

We omit subscripts whenever the meaning is clear.

[1] A set X is countable if there is an injection from X to the natural numbers.

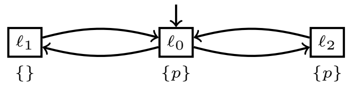


Figure 1: A Kripke structure on $AP = \{p\}$, with an initial state ℓ_0. The labels $\lambda(\ell_i)$ are shown below each state ℓ_i.

2.1. Linear Temporal Logic

We recall the definition of *linear temporal logic* [11], LTL for short. For the remainder of the paper, we fix an arbitrary finite, nonempty set AP of atomic propositions.

Definition 1 (LTL Formulae)**.** *Let p range over AP. The* formulae of LTL *are generated by the following grammar:*

$$\varphi ::= \bot \mid p \mid \neg\varphi \mid \varphi \vee \varphi \mid \mathbf{X}\,\varphi \mid \varphi\,\mathbf{U}\,\varphi$$

Fm_{LTL} denotes the set of all LTL formulae.

In LTL, time is interpreted as a linear timeline that unfolds infinitely into the future. The operator $\mathbf{X}$ states that a formula holds in the *next* time step, while $\varphi\,\mathbf{U}\,\psi$ means that φ holds *until* ψ holds (and ψ does eventually hold). We define the usual abbreviations for boolean operations ($\top$, $\wedge$, $\to$) as well as the temporal operators $\mathbf{F}\,\varphi := \top\,\mathbf{U}\,\varphi$ (*finally*, at some point in the future), $\mathbf{G}\,\varphi := \neg\mathbf{F}\,\neg\varphi$ (*globally*, at all points in the future), and $\mathbf{X}^k\,\varphi$ for repeated application of $\mathbf{X}$, where $k \in \mathbb{N}$.

Formally, timelines are modelled as *traces*. A trace is an infinite sequence $\pi = a_0 a_1 \ldots$, where each $a_i \in \mathcal{P}(AP)$ is a set of atomic propositions that hold at time step i. The infinite suffix of π starting at time step i is denoted by $\pi^i = a_i a_{i+i} \cdots$. The set of all traces is denoted by $\mathcal{P}(AP)^\omega$.

The semantics of LTL is defined in terms of Kripke structures [11], which describe possible traces.

Definition 2 (Kripke Structure)**.** *A* Kripke structure *is a tuple $M = (S, I, T, \lambda)$ such that S is a finite set of states; $I \subseteq S$ is a non-empty set of initial states; $T \subseteq S \times S$ is a left-total transition relation, i.e., for all $s \in S$ there exists $s' \in S$ such that $(s, s') \in T$; and $\lambda : S \to \mathcal{P}(AP)$ labels states with sets of atomic propositions.*

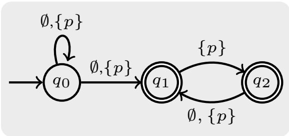
A trace of a Kripke structure M is a sequence $\pi = \lambda(s_0)\lambda(s_1)\lambda(s_2)\ldots$ with $s_0 \in I$, and for all $i \geq 0$, $s_i \in S$ and $(s_i, s_{i+1}) \in T$. The set of all traces from a Kripke structure M is given by $Traces(M)$. Figure 1 shows an example of a Kripke structure, in graphical notation.

The satisfaction relation between Kripke structure and LTL formulae is defined in terms of the satisfaction between the Kripke structure's traces and LTL formulae.

Definition 3 (Satisfaction)**.** *The* satisfaction relation *between traces and LTL formulae is the least relation $\models\ \subseteq \mathcal{P}(AP)^\omega \times Fm_{LTL}$ such that, for all traces $\pi = a_0 a_1 \ldots \in \mathcal{P}(AP)^\omega$:*

$$\begin{array}{lll}
\pi \not\models \bot & & \\
\pi \models p & \textit{iff} & p \in a_0 \\
\pi \models \neg\varphi & \textit{iff} & \pi \not\models \varphi \\
\pi \models \varphi_1 \vee \varphi_2 & \textit{iff} & \pi \models \varphi_1 \textit{ or } \pi \models \varphi_2 \\
\pi \models \mathbf{X}\,\varphi & \textit{iff} & \pi^1 \models \varphi \\
\pi \models \varphi_1\,\mathbf{U}\,\varphi_2 & \textit{iff} & \textit{there exists } i \geq 0 \textit{ s.t. } \pi^i \models \varphi_2 \\
 & & \textit{and for all } j < i, \pi^j \models \varphi_1
\end{array}$$

Büchi automaton $A_{\mathcal{K}}$:



Some Infinite Words from $\mathcal{L}(A_{\mathcal{K}})$:

$\pi_1 = \emptyset\,\emptyset\,\emptyset\,\{p\}\,(\emptyset\,\{p\})^\omega$

$\pi_2 = \{p\}\,\{p\}\,\emptyset\,(\emptyset\,\{p\})^\omega$

$\pi_3 = \{p\}\,\{p\}\,\emptyset\,\{p\}^\omega$

Figure 2: A Büchi automaton (on the right), and some infinite words accepted by this automaton (on the left).

A Kripke structure M satisfies a formula φ, denoted $M \models \varphi$, iff all traces of M satisfy φ. M satisfies a set X of formulae, $M \models X$, iff $M \models \varphi$ for all $\varphi \in X$. The consequence operator Cn_{LTL} is defined from the satisfaction relation.

Definition 4 (Consequence Operator)**.** *The* consequence operator Cn_{LTL} *maps each set* X *of LTL formulae to the set of all formulae* ψ*, such that for all Kripke structures* M*, if* $M \models X$ *then also* $M \models \psi$*.*

Observation 5. *LTL is Tarskian and Boolean.*

2.2. Büchi Automata

Büchi automata are finite automata widely used in formal specification and verification of systems, specially in LTL model checking [11]. Büchi automata have also been used in planning to synthesize plans when goals are in LTL [26, 30].

Definition 6 (Büchi Automata)**.** *A* Büchi automaton *is a tuple* $A = (Q, \Sigma, \Delta, Q_0, R)$ *consisting of a finite set of states* Q*; a finite, nonempty alphabet* Σ *(whose elements are called* letters*); a transition relation* $\Delta \subseteq Q \times \Sigma \times Q$*; a set of initial states* $Q_0 \subseteq Q$*; and a set of* recurrence states $R \subseteq Q$*.*

We show a concrete Büchi automaton in Example 7.

A Büchi automaton accepts an infinite word over a finite alphabet Σ, if the automaton visits a recurrence state infinitely often while reading the word. Formally, an infinite word is a sequence $a_0a_1\ldots$ with $a_i \in \Sigma$ for all i. For a finite word $\rho = a_0 \ldots a_n$, with $n \geq 0$, let ρ^ω denote the infinite word corresponding to the infinite repetition of ρ. The set of all infinite words is denoted by Σ^ω. An infinite word $a_0a_1a_2 \ldots \in \Sigma^\omega$ is *accepted* by a Büchi automaton $A = (Q, \Sigma, \Delta, Q_0, R)$ if there exists a sequence $q_0, q_1, q_2, \ldots$ of states $q_i \in Q$ such that $q_0 \in Q_0$ is an initial state, for all i we have that $(q_i, a_i, q_{i+1}) \in \Delta$ and there are infinitely many $i \in \mathbb{N}$ with $q_i \in R$. The set $\mathcal{L}(A)$ of all accepted words is the *language* of A.

In this work, unless otherwise noted, we always consider Büchi automata over the alphabet $\Sigma = \mathcal{P}(AP)$, where letters are sets of atomic propositions and infinite words are traces. The following example presents such an automaton.

Example 7. *Figure 2 illustrates (on the left) a Büchi automaton* $A_{\mathcal{K}}$ *over the alphabet* $\Sigma = \{\emptyset, \{p\}\}$*. States are depicted as circles and each transition* (q, a, q') *is depicted as an arrow from* q *to* q' *labelled with* a*. The initial state is* q_0*, and the recurrence states are marked as double circles, i. e.,* q_1 *and* q_2*.*

The right-hand side of Figure 2 shows some of the infinite words accepted by the Büchi automaton $A_{\mathcal{K}}$*. Consider, for instance, the sequence*

$$q_0 \xrightarrow{\emptyset} q_0 \xrightarrow{\emptyset} q_0 \xrightarrow{\emptyset} q_1 \xrightarrow{\{p\}} q_2 \xrightarrow{\emptyset} q_1 \xrightarrow{\{p\}} q_2 \ldots,$$

where each arrow $q \xrightarrow{x} q'$ *indicates the transition* (q, x, q') *in the automaton. By concatenating the letters in this sequence, we get the infinite word* π_1 *defined in Figure 2. The acceptance condition requires some recurrence states to appear infinitely often. As for instance the recurrence state* q_1 *appears infinitely often, the acceptance condition holds and* π_1 *is accepted. Analogously, the infinite words* π_2 *and* π_3 *are also accepted.*

On the other hand, the infinite word $\pi' = \emptyset^\omega$ *is not accepted, as the only sequence of states that produces this word is* $q_0 \xrightarrow{\emptyset} q_0 \xrightarrow{\emptyset} q_0 \ldots$*, where* $q_0 \xrightarrow{\emptyset}$ *loops. The only state in this sequence is* q_0 *which is not a recurrence state and, therefore, the acceptance condition is violated.*

Emptiness of a Büchi automaton's language is decidable. Further, Büchi automata for the union, intersection and complement of the languages of given Büchi automata can be effectively constructed [23]. In the remainder of the paper we specifically make use of the construction for union and intersection, and denote them with the symbol $\sqcup$ resp. $\sqcap$. The automata-theoretic treatment for several crucial reasoning problems in LTL, such as model-checking and satisfiability, is based on the following result:

Proposition 8 ([11])**.** *For every LTL formula* φ *and every Kripke structure* M*, there exist Büchi automata* A_φ *and* A_M *that accept precisely the traces that satisfy* φ *resp. the traces of* M*, that is,* $\mathcal{L}(A_\varphi) = \{\pi \in \mathcal{P}(AP)^\omega \mid \pi \models \varphi\}$*, and* $\mathcal{L}(A_M) = Traces(M)$*.*

The proposition above states that every LTL formula φ can be expressed as a Büchi automaton A_φ, in the sense that A_φ accepts exactly all the traces satisfying φ. This result allows to decide if a formula φ is satisfiable, by deciding emptiness of $\mathcal{L}(A_\varphi)$. Analogously, a Büchi automaton A_M can be used to capture precisely all the traces from a given Kripke structure M, as Proposition 8 states. These two observations make it possible to decide LTL model-checking, by deciding the inclusion $\mathcal{L}(A_M) \subseteq \mathcal{L}(A_\varphi)$. In Section 5, we will exploit Proposition 8 to devise mechanisms that support the computation of belief change operators in LTL.

3. AGM Contraction

In the AGM paradigm, the epistemic state of an agent is represented as a theory. A contraction function for a theory $\mathcal{K}$ is a function $\dot{-} : Fm \to \mathcal{P}(Fm)$ that given an unwanted piece of information φ outputs a subset of $\mathcal{K}$ which does not entail φ. Contraction functions are subject to the following rationality postulates [2]:

$(\mathbf{K_1^-})$	$\mathcal{K} \dot{-} \varphi = Cn(\mathcal{K} \dot{-} \varphi)$	(closure)
$(\mathbf{K_2^-})$	$\mathcal{K} \dot{-} \varphi \subseteq \mathcal{K}$	(inclusion)
$(\mathbf{K_3^-})$	If $\varphi \notin \mathcal{K}$, then $\mathcal{K} \dot{-} \varphi = \mathcal{K}$	(vacuity)
$(\mathbf{K_4^-})$	If $\varphi \notin Cn(\emptyset)$, then $\varphi \notin \mathcal{K} \dot{-} \varphi$	(success)
$(\mathbf{K_5^-})$	$\mathcal{K} \subseteq (\mathcal{K} \dot{-} \varphi) + \varphi$	(recovery)
$(\mathbf{K_6^-})$	If $\varphi \equiv \psi$, then $\mathcal{K} \dot{-} \varphi = \mathcal{K} \dot{-} \psi$	(extensionality)
$(\mathbf{K_7^-})$	$(\mathcal{K} \dot{-} \varphi) \cap (\mathcal{K} \dot{-} \psi) \subseteq \mathcal{K} \dot{-} (\varphi \wedge \psi)$	
$(\mathbf{K_8^-})$	If $\varphi \notin \mathcal{K} \dot{-} (\varphi \wedge \psi)$ then $\mathcal{K} \dot{-} (\varphi \wedge \psi) \subseteq \mathcal{K} \dot{-} \varphi$	

For a detailed discussion on the rationale of these postulates, see [1, 2, 3]. The postulates $(\mathbf{K_1^-})$ to $(\mathbf{K_6^-})$ are called the basic rationality postulates, while $(\mathbf{K_7^-})$ and $(\mathbf{K_8^-})$ are known as supplementary postulates. A contraction function that satisfies the basic rationality postulates is called a rational contraction function. If a contraction function satisfies all the eight rationality postulates, we say that it is fully rational.

There are many different constructions for (fully) rational AGM contraction such as Partial Meet [1], Epistemic Entrenchment [2], and Kernel Contraction [6]. All these functions are characterized by the AGM postulates of contraction. For an overview, see [31, 3]. These contraction functions, however, are rational only in very specific logics, precisely in the presence of the AGM assumptions [8] which includes requiring the logic to be Boolean and compact. See [8] for details about the AGM assumptions.

To embrace more expressive logics, Ribeiro et al. [18] have proposed a new class of (fully) rational contraction functions which only assume the underlying logic to be Tarskian and Boolean: the Exhaustive Contraction Functions (for basic rationality) and the Blade Contraction Functions (for full rationality). We briefly review Exhaustive Contraction Functions. We do not delve into the Blade Contraction Functions, as our results for full rationality do not use such functions, but rather use the supplementary postulates directly.

Definition 9 (Choice Functions)**.** *A* choice function *is a function* $\delta : Fm \to \mathcal{P}(CCT)$ *maps each formula* φ *to a set of complete consistent theories satisfying the following:*

(CF1) $\delta(\varphi) \neq \emptyset$*;*

(CF2) *if* $\varphi \notin Cn(\emptyset)$*, then* $\delta(\varphi) \subseteq \overline{\omega}(\varphi)$*; and*

(CF3) *for all* $\varphi, \psi \in Fm$*, if* $\varphi \equiv \psi$ *then* $\delta(\varphi) = \delta(\psi)$*.*

A choice function chooses at least one complete consistent theory, for each formula φ to be contracted **(CF1)**. As long as φ is not a tautology, the CCTs chosen must not contain the formula φ **(CF2)**, since the goal is to relinquish φ. The last condition **(CF3)** imposes that a choice function is syntax-insensitive.

Definition 10 (Exhaustive Contraction Functions)**.** *Let* δ *be a choice function. The* Exhaustive Contraction Function (ECF) *on a theory* $\mathcal{K}$ *induced by* δ *is the function* $\dot{-}_\delta$ *such that* $\mathcal{K} \dot{-}_\delta \varphi = \mathcal{K} \cap \bigcap \delta(\varphi)$*, if* $\varphi \notin Cn(\emptyset)$ *and* $\varphi \in \mathcal{K}$*; otherwise,* $\mathcal{K} \dot{-}_\delta \varphi = \mathcal{K}$*.*

Whenever the formula φ to be contracted is not a tautology and is in the theory $\mathcal{K}$, an ECF modifies the current theory by selecting some CCTs and intersecting them with $\mathcal{K}$. On the other hand, if φ is either a tautology or is not in the theory $\mathcal{K}$, then all beliefs are preserved. The ECFs are similar in spirit to the standard *partial meet* functions [1]. The main difference is that partial meet relies on the internal structure of the current theory by selecting and intersecting remainders (maximal non-entailing subsets), whilst ECF chooses external structures (CCTs). In the latter, CCTs are used, because, in the absence of compactness, remainders do not exist in general [12, 18, 32].

Theorem 11. *[18] A contraction function* $\dot{-}$ *is rational iff it is an ECF.*

4. Limits of Finite Representation

In the AGM paradigm, the epistemic states of an agent are represented as theories which are in general infinite. However, according to Hansson [33, 34], the epistemic states of rational agents should have a finite representation. This is motivated from the perspective that epistemic states should resemble the cognitive states of human minds, and Hansson argues that as "finite beings", humans cannot sustain epistemic states that do not have a finite representation. Further, finite representation is crucial from a computational perspective, to represent epistemic states in a computer.

Different strategies of finite representation have been used such as (i) finite bases [35, 36, 37], and (ii) finite sets of models [38, 39]. In the former strategy, each finite set X of formulae, called a *finite base*, represents the theory $Cn(X)$. In the latter strategy, models are used to represent an epistemic state. Precisely, each finite set X of models represents the theory of all formulae satisfied by all models in X, that is, the theory $\{\varphi \in Fm_{\mathbb{L}} \mid M \models \varphi, \text{ for all } M \in X\}$. The expressiveness of finite bases and finite sets of models are, in general (depending on the logic), incomparable, that is, some theories expressible in one method cannot be expressed in the other method and vice versa. For instance, the information that *"John swims every two days"* cannot be expressed via a finite base in LTL [40], although it can be expressed via a single Kripke structure (shown in Fig. 1, where p stands for *"John swims"*). On the other hand, *"Anna will swim eventually"* is expressible as a single LTL formula ($\mathbf{F}\, s$, where s stands for *"Anna swims"*), but it cannot be expressed via a finite set of models.

Given the incomparable expressiveness of these two well-established strategies of finite representations, it is not clear whether in general, and specifically in non-finitary logics, there exists a method capable of finitely representing all theories, therefore capturing the whole expressiveness of the logic. Towards answering this question, we provide a broad definition to conceptualise finite representation.

A finite representation for a theory can been seen as a finite word, i.e., a *code*, from a fixed finite alphabet Σ_C. For example, the codes $c_1 := \{\mathtt{a},\ \mathtt{b}\}$ and $c_2 := \{\mathtt{a},\ \mathtt{a}\to\mathtt{b}\}$ are finite words in the language of set theory, and both represent the theory $Cn(\{a \wedge b\})$. The set of all codes, i.e., of all finite words over Σ_C, is denoted by Σ_C^*. In this sense, a method of finite representation is a mapping f from codes in Σ_C^* to theories. The pair (Σ_C, f) is called an *encoding*.

Definition 12 (Encoding)**.** *An* encoding (Σ_C, f) *comprises a finite alphabet* Σ_C *and a partial function* $f : \Sigma_C^* \rightharpoonup \mathsf{Th}_{\mathbb{L}}$*.*

Given an encoding (Σ_C, f), a word $w \in \Sigma_C^*$ represents a theory $\mathcal{K}$, if $f(w)$ is defined and $f(w) = \mathcal{K}$. Observe that a theory might have more than one code, whereas for others there might not exist a code. For instance, in the example above for finite bases, the codes c_1 and c_2 represent the same theory. On the other hand, recall that the LTL theory corresponding to *"John swims every two days"* cannot be expressed in the finite base encoding. Furthermore, the function f is partial, because not all codes in Σ_C^* are meaningful. For instance, for the finite base encoding, the code $\{\{\}\}$ cannot be interpreted as a finite base.

We are interested in logics which are AGM compliant, that is, logics in which rational contraction functions exist. Unfortunately, it is still an open problem how to construct AGM contraction functions in all such logics. The most general constructive apparatus up to date, as discussed in Section 3, are the Exhaustive Contraction functions proposed by Ribeiro et al. (2018) which assume only few conditions on the logic. Additionally, we focus on non-finitary logics, as the finitary case is trivial. We call such logics *compendious*.

Definition 13 (Compendious Logics)**.** *A logic* $\mathbb{L}$ *is* compendious *if* $\mathbb{L}$ *is Tarskian, Boolean, non-finitary and satisfies:*

(Discerning) *For all sets $X, Y \subseteq CCT_{\mathbb{L}}$, we have that $\bigcap X = \bigcap Y$ implies $X = Y$.*

Compendiousness amounts to expressivity in multiple dimensions. Compendious logics can express infinitely many distinct sentences (non-finitary), distinguish between a sentence being true or false (classical negation), and express uncertainty of two or more sentences (disjunction). The property **(Discerning)** is related the possible worlds semantics. In a possible world, the truth values of all sentences are known. From this perspective, possible worlds correspond to CCTs. Under the possible worlds semantics, an agent's epistemic state is interpreted as the exact set of all possible worlds in which all the agent's beliefs are true. If the truth value of a formula φ is unknown, the agent considers some possible worlds where φ is true, as well as possible worlds where φ is false. Hence, more possible worlds indicate strictly less information. Equivalently, different sets of possible worlds represent different epistemic states. This is exactly what **(Discerning)** conceptualises.

Example 14 (Discerning)**.** *Yara and Yasmin encounter a large flightless bird. Yara knows that such birds exist in Africa and South America. Hence, Yara considers two possible worlds: the bird is from Africa (it is an ostrich), or the bird is from South America (it is a rhea). Yasmin, who lived in Australia, believes the bird is an emu (from Australia), a rhea or an ostrich. Since Yara and Yasmin consider different sets of possible worlds, their epistemic states differ. Yara believes in the disjunction $ostrich \vee rhea$, Yasmin does not. She believes only in the disjunction $ostrich \vee rhea \vee emu$. As per **(Discerning)**, Yara and Yasmin present different epistemic states, due to the difference in the considered possible worlds.*

The class of compendious logics is broad and includes several widely used logics.

Theorem 15. *The logics LTL, CTL, CTL*, μ-calculus and* monadic second-order logic *(MSO) are compendious.*

It turns out that there is no method of finite representation capable of capturing all theories in a compendious logic.

Theorem 16. *No encoding can represent every theory of a compendious logic.*

Proof Sketch. We show that, since compendious logics are Tarskian, Boolean and non-finitary, there exist infinitely many CCTs. From **(Discerning)**, it follows that there exist uncountably many theories in the logic. However, an encoding can represent only countably many theories. □

As not every theory cann be finitely represented, only some subsets of theories can be used to express the epistemic states of an agent. We call a subset $\mathbb{E}$ of theories an *excerpt* of the logic. Each encoding induces an excerpt.

Definition 17. *The excerpt* induced *by an encoding (Σ_{C}, f) is the set $\mathbb{E} := \mathsf{img}(f)$. An excerpt induced by some encoding is called* finitely representable.

5. The Büchi Encoding of LTL

The encoding in which epistemic states are expressed is of fundamental importance. On the one hand, the encoding must be expressive enough to capture a non-trivial space of epistemic states. On the other hand, the encoding must support reasoning. Most fundamentally, an agent should be able to decide whether it believes a given formula φ, i.e., whether φ is entailed by the theory representing the agent's epistemic state. This question might be decidable for one (perhaps less expressive) encoding, but undecidable for a different (more expressive) encoding. We call this question the *entailment problem on an encoding* (Σ_{C}, f):

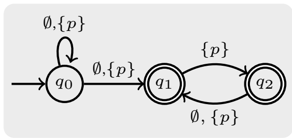
Input: $(w, \varphi) \in \Sigma_{\mathrm{C}}^* \times Fm_{\mathbb{L}}$, such that $f(w)$ is defined

Output: *true* if $\varphi \in f(w)$, otherwise *false*

This problem is a generalisation of several decision problems that support reasoning. For example, on the finite base encoding, it corresponds to the usual entailment problem between formulae. Entailment on the encoding based on finite sets of models corresponds to the model checking problem. For other encodings, as we will see, it can be more general than either of these problems.

We investigate a suitable encoding for epistemic states over LTL, a commonly used compendious logic in model checking and planning. In both these domains, the primary approach to reason about LTL is based on Büchi automata. Thus, from a reasoning standpoint, Büchi automata are predestined to be the basis for an encoding of epistemic states over LTL. We give the following definition for the set of LTL formulae represented by a Büchi automaton:

Büchi automaton $A_{\mathcal{K}}$:



Supported Formulae:

$\mathbf{F}\, p \in \mathcal{S}(A_{\mathcal{K}})$

$\mathbf{G}\,\mathbf{F}\, p \in \mathcal{S}(A_{\mathcal{K}})$

$\mathbf{F}\,(p \rightarrow \mathbf{X}\, p \vee \mathbf{X}^2\, p) \in \mathcal{S}(A_{\mathcal{K}})$

$\mathbf{G}\, p, \neg(\mathbf{G}\, p) \notin \mathcal{S}(A_{\mathcal{K}})$

Figure 3: A Büchi automaton, along with some examples of supported (and not supported) LTL formulae.

Definition 18 (Support)**.** *The* support *of a Büchi automaton A is the set $\mathcal{S}(A) := \{\, \varphi \in Fm_{LTL} \mid \forall \pi \in \mathcal{L}(A) \,.\, \pi \models \varphi \,\}$. If $\varphi \in \mathcal{S}(A)$, we say that A* supports *φ.*

Example 19. *Figure 3 shows a Büchi automaton (on the left), along with three supported formulae (on the right). All accepted traces satisfy these formulae. The formula $\mathbf{G}\, p$ is not supported. While some accepted traces, such as $\{p\}^\omega$, satisfy this formula, others, such as $\emptyset\,\{p\}^\omega$ do not. Consequently, the negation $\neg(\mathbf{G}\, p)$ is not supported either.*

It remains to show that the support of a Büchi automaton is a theory. Perhaps surprisingly, the support of an arbitrary language of infinite traces (not represented as a Büchi automaton) does not necessarily form a theory. The disconnect arises from the fact that the semantics of LTL is defined over finite Kripke structures, and arbitrary languages of infinite traces can represent more fine-grained nuances of behaviours. Consider the language $L_{\mathsf{prime}} = \{a_0 a_1 a_2 \ldots\}$, where $a_i = \{p\}$ if i is a prime number and $a_i = \emptyset$ otherwise. The support of L_{prime} prescribes that p holds exactly in prime-numbered steps. Since no Kripke structure satisfies this requirement, the support of L_{prime} is inconsistent, yet L_{prime} does not support $\bot$.

An intriguing property of Büchi automata however is that their support is fully determined by those accepted traces π that have the property of being *ultimately periodic*, that is, $\pi = \rho\,\sigma^\omega$ for some finite sequences ρ, σ. Recall from Section 2.2 that the superscript ω denotes infinite repetition of the subsequence σ. Ultimately periodic traces are tightly connected to CCTs: each CCT is satisfied by exactly one ultimately periodic trace. Let UP denote the set of all ultimately periodic traces. The correspondence between CCTs and ultimately periodic traces is formalized by the function $Th_{UP} : UP \to CCT_{LTL}$ such that $Th_{UP}(\pi) = \{\varphi \in Fm_{LTL} \mid \pi \models \varphi\}$.

Lemma 20. *The function Th_{UP} is a bijection.*

We combine Lemma 20 with two classical observations [11]: (i) every consistent LTL formula is satisfied by at least one ultimately periodic trace; and (ii) every Büchi automaton with nonempty language accepts some ultimately periodic trace. We arrive at the following characterization:

Lemma 21. *The support of a Büchi automaton A satisfies*

$$\mathcal{S}(A) = \bigcap \{ \, Th_{UP}(\pi) \mid \pi \in \mathcal{L}(A) \cap UP \, \} .$$

Theorem 22. *The support of a Büchi automaton is a theory.*

Thus, Büchi automata indeed define an encoding. Every Büchi automaton A, being a finite structure, can be described in a finite code word w_A, which the encoding maps to the theory $\mathcal{S}(A)$. We call this encoding the *Büchi encoding*, denoted $(\Sigma_{\text{Büchi}}, f_{\text{Büchi}})$, and the induced excerpt the *Büchi excerpt* $\mathbb{E}_{\text{Büchi}}$. In terms of expressiveness, the Büchi excerpt strictly subsumes the classical approaches:

Theorem 23. *Let $\mathbb{E}_{\text{base}}$ and $\mathbb{E}_{\text{models}}$ denote respectively the excerpts of finite bases and finite sets of models[2]. It holds that $\mathbb{E}_{\text{base}} \cup \mathbb{E}_{\text{models}} \subsetneq \mathbb{E}_{\text{Büchi}}$.*

Proof Sketch. The expressiveness of the Büchi excerpt follows from Proposition 8. Figure 3 shows an automaton whose support can be expressed neither by a finite base nor a finite sets of models. □

In terms of reasoning, the Büchi encoding benefits from the decidability properties of Büchi automata. Many decision problems, most importantly the entailment problem on the Büchi encoding, can be reduced to the decidable problem of inclusion between Büchi automata.

Theorem 24. *The entailment problem on the Büchi encoding is decidable.*

Proof. Given a word $w \in \Sigma^*_{\text{Büchi}}$ that encodes a Büchi automaton A_w, and an LTL formula φ, one can decide whether $\varphi \in f_{\text{Büchi}}(w) = \mathcal{S}(A_w)$ by deciding the Büchi automata inclusion $\mathcal{L}(A_w) \subseteq \mathcal{L}(A_\varphi)$. □

Beyond ensuring the decidability of key problems, an encoding's suitability for reasoning also involves the question whether modifications of epistemic states can be realized by computations on code words. In particular in the context of the AGM paradigm, it is interesting to see if belief change operations can be performed in such a manner. The Büchi encoding also shines in this respect, since we can employ automata operations to this end. As a first example, consider the *expansion* of a theory $\mathcal{K}$ with a formula φ. This operation corresponds to an intersection operation on Büchi automata, as the support of a Büchi automaton satisfies $\mathcal{S}(A) + \varphi = \mathcal{S}(A \sqcap A_\varphi)$. The intersection automaton $A \sqcap A_\varphi$ can be computed through a product construction.

[2]These excerpts were described in the prologue of Section 4.

By contrast, the following two sections discuss fundamental limitations to effective constructions for rational *contractions*. Nevertheless, we show in Section 8 how the Büchi encoding admits similar automata-based constructions for a large subclass of contraction functions.

6. AGM Accommodation

Assume that the space of epistemic states that an agent can entertain is determined by an excerpt $\mathbb{E}$. In this section, we investigate which properties make an excerpt suitable from the AGM vantage point. Clearly, not every excerpt is suitable for representing the space of epistemic states. For example, if a non-tautological formula φ appears in each theory of $\mathbb{E}$, then φ cannot be contracted. The chosen excerpt should be expressive enough to describe all relevant epistemic states that an agent might hold in response to its beliefs in flux. Precisely, if an agent is confronted with a piece of information and changes its epistemic state into a new one, then this new epistemic state must be expressible in the underlying excerpt. A straightforward option would be to require some sort of closure under AGM rationality, that is, all possible rational contractions involving information in the excerpt should be expressed yet within the excerpt. Such excerpts are *closed under rational contraction* resp. *under fully rational contraction.* We say that a contraction $\dot{-}$ *remains in* $\mathbb{E}$ if $\mathsf{img}(\dot{-}) \subseteq \mathbb{E}$.

Definition 25 (Closedness)**.** *An excerpt $\mathbb{E}$ of a logic $\mathbb{L}$ is* closed under (fully) rational contraction *iff for all theories $\mathcal{K} \in \mathbb{E}$, every (fully) rational contraction operation on $\mathcal{K}$ remains in $\mathbb{E}$.*

Closedness maximises the expressiveness of the excerpt w.r.t. AGM rationality: in each epistemic state of the excerpt, every possible (fully) AGM rational contraction outcome is at disposal. However, although closedness might seem like a reasonable condition, it turns out to be very demanding. For example, as we are dealing with Boolean logics, which are closed under classical negation, an agent should be able to either accept or reject some pieces of information. The excerpt should be broad enough such that there exists some piece of information φ, where both φ and $\neg\varphi$ occur in some, possible different, epistemic states of the excerpt. We call such excerpts *open-minded.* Even under such an innocuous condition, an agent cannot express its epistemic states in an excerpt that is closed under rational contraction: closedness rules out finite representability.

Theorem 26 (Impossibility of Closedness)**.** *If $\mathbb{E}$ is an open-minded, finitely representable excerpt of a compendious logic, then $\mathbb{E}$ is not closed under rational contraction.*

Proof Sketch. From open-mindedness, it follows that there exists a formula φ in a theory $\mathcal{K}$ of the excerpt, such that $\overline{\omega}(\varphi)$ is infinite. Then there are already uncountably many ways to contract φ. However, the finitely representable excerpt contains only countably many theories. □

The negative result above concerns excerpts that are closed under rational contraction. As full rationality is strictly more demanding than rationality, one could hope to reach closedness by restricting to excerpts closed under fully rational contraction. Unfortunately, rationally closed excerpts and fully rationally closed excerpts coincide.

Proposition 27. *An excerpt is closed under fully rational contraction iff it is closed under rational contraction.*

Instead of insisting on maximising the expressiveness of the excerpts, we impose a weaker condition and require the excerpt only to admit at least one rational outcome for each possible contraction.

Definition 28 (Accommodation)**.** *An excerpt* $\mathbb{E}$ accommodates (fully) rational contraction *iff for each* $\mathcal{K} \in \mathbb{E}$ *there exists a (fully) rational contraction on* $\mathcal{K}$ *that remains in* $\mathbb{E}$.

Accommodation guarantees that an agent can modify its beliefs rationally, in all possible epistemic states covered by the excerpt. There is a clear connection between accommodation and AGM compliance. While AGM compliance concerns existence of rational contraction operations in every theory of a logic, accommodation guarantees that the information in each theory within the excerpt can be rationally contracted and that its outcome can yet be expressed within the excerpt. Analogous to closedness, rational accommodation and fully rational accommodation coincide.

Proposition 29. *An excerpt* $\mathbb{E}$ *accommodates rational contraction iff* $\mathbb{E}$ *accommodates fully rational contraction.*

7. Uncomputability of Contraction

Accommodation is the weakest condition we can impose upon an excerpt to guarantee the existence of AGM rational contractions. Yet, the existence of contractions does not imply that an agent can *effectively* contract information. Thus we investigate the question of *computability* of contraction functions. For this endeavor, the focus on contraction functions that remain in the excerpt is crucial: both input and output of a computation must be finitely representable. We thus fix a finitely representable excerpt $\mathbb{E}$ that accommodates contraction. As an agent has to reason about its beliefs, it should be able to decide whether two formulae are logically equivalent. Hence, we assume that, in the underlying logic, logical equivalence is decidable.

Definition 30 (AGM Computability)**.** *Let* $\mathcal{K}$ *be a theory in* $\mathbb{E}$*, and let* $\dot{-}$ *be a contraction function on* $\mathcal{K}$ *that remains in* $\mathbb{E}$*. We say that* $\dot{-}$ *is* computable *if there exists an encoding* (Σ_{C}, f) *that induces* $\mathbb{E}$*, such that the following problem is computed by a Turing machine:*

> ***Input:*** *A formula* $\varphi \in Fm_{\mathbb{L}}$.
>
> ***Output:*** *A word* $w \in \Sigma_{\mathrm{C}}^*$ *such that* $f(w) = \mathcal{K} \dot{-} \varphi$.

In the classical setting of finitary logics, computability of AGM contraction is trivial, as there are only finitely many formulae (up to equivalence), and only a finite number of theories. By contrast, compendious logics have infinitely many formulae (up to equivalence) and consequently infinitely many theories.

In the following, unless otherwise stated, we only consider compendious logics. In such logics, we distinguish two kinds of theories: those that contain infinitely many formulae (up to equivalence), and those that contain only finitely many formulae (up to equivalence). An excerpt that constrains an agent's epistemic states to the latter case essentially disposes of the expressive power of the compendious logic, as in each epistemic state only finitely many sentences can be distinguished. Therefore, such epistemic states could be expressed in a finitary logic. As the computability in the finitary case is trivial, we focus on the more expressive case.

Definition 31 (Non-Finitary)**.** *A theory* $\mathcal{K}$ *is* non-finitary *if it contains infinitely many logical equivalence classes of formulae.*

Note that being non-finitary is a very general condition. Even theories with a finite base can be non-finitary. For instance, the LTL theory $Cn(\mathbf{G}\, p)$ contains the infinitely many non-equivalent formulae $\{p, \mathbf{X}\, p, \mathbf{X}^2\, p, \mathbf{X}^3\, p, \ldots\}$.

In the remainder of this section, we establish a strong link between non-finitary theories and uncomputable contraction functions. To this end, we introduce the notion of *cleavings.*

Definition 32 (Cleaving)**.** *A* cleaving *is an infinite set of formulae* $\mathcal{C}$ *such that for all two distinct* $\varphi, \psi \in \mathcal{C}$ *we have:*

(CL1) *φ and ψ are not equivalent ($\varphi \not\equiv \psi$); and*

(CL2) *the disjunction $\varphi \vee \psi$ is a tautology.*

From an algebraic perspective, the formulae in a cleaving behave like a kind of weak complement: we require that the disjunction $\varphi \vee \psi$ is a tautology, whereas we do not require the conjunction $\varphi \wedge \psi$ to be inconsistent (as would be the case for the conjunction $\varphi \wedge \neg\varphi$).

Example 33. *Consider the logic of elementary arithmetic over natural numbers. The formulae $x \neq 0$, $x \neq 1$, $x \neq 2$, etc. form a cleaving: they are pairwise non-equivalent, and every disjunction $(x \neq i) \vee (x \neq j)$ is a tautology (where i, j are two different constants).*

Example 34. *Let $twice(p) :\equiv \mathbf{F}\,(p \wedge \mathbf{X}\,\mathbf{F}\, p)$ be the LTL formula denoting that proposition p holds at least twice, and for $i \in \mathbb{N}$, let $\psi_i :\equiv (\mathbf{X}^i\, p) \to twice(p)$ be the formula stating that if p holds in time step i, it must hold at least one additional time (i.e., at least twice overall). For $i \neq j$, the formulae ψ_i and ψ_j are non-equivalent. Further, the disjunction $\psi_i \vee \psi_j$ simplifies to $(\mathbf{X}^i\, p) \wedge (\mathbf{X}^j\, p) \to twice(p)$. The latter formula is a tautology: if p holds in time steps i and j, it holds at least twice. Hence, the set of formulae $\{\psi_0, \psi_1, \ldots\}$ is a cleaving.*

Lemma 35. *Every non-finitary theory contains a cleaving.*

Given a contraction that remains in an excerpt, cleavings provide a way of generating many different contractions that remain within the excerpt. This works by ranking the formulae in the cleaving such that each rank has exactly one formula. We reduce the contraction of a formula φ to contracting $\varphi \vee \psi$, where ψ is the lowest ranked formula in the cleaving such that $\varphi \vee \psi$ is non-tautological. Each new contraction depends on the original choice function and the ranking.

Definition 36 (Composition)**.** *Let δ be a choice function on a theory $\mathcal{K}$, $\mathcal{C} \subseteq \mathcal{K}$ be a cleaving, and $\pi : \mathbb{N} \to \mathcal{C}$ be a permutation of $\mathcal{C}$. The* composition of δ and π *is the function $\delta_\pi : Fm \to \mathcal{P}(CCT)$ such that*

$$\delta_\pi(\varphi) := \delta\big(\varphi \vee \min_\pi(\varphi)\big)$$

where $\min_\pi(\varphi) = \pi(i)$, for the minimal $i \in \mathbb{N}$ such that $\varphi \vee \pi(i) \not\equiv \top$, or $\min_\pi(\varphi) = \bot$ if no such i exists.

Example 37. *Consider the cleaving $\{\psi_0, \psi_1, \ldots\}$ of Example 34, and let π be the permutation with $\pi(i) = \psi_i$ for all $i \in \mathbb{N}$. The formula $p \vee \psi_0$ is a tautology. Thus, we have $\min_\pi(p) = \pi(1) = \psi_1$ and the choice function chooses $\delta_\pi(p) = \delta(p \vee \psi_1)$. If we however consider a permutation π' with $\pi'(0) = \psi_2$ and $\pi'(2) = \psi_0$, then we have $\min_{\pi'}(p) = \pi'(0) = \psi_2$ and $\delta_{\pi'}(p) = \delta(p \vee \psi_2)$.*

If we consider the formula $\mathbf{F}\,\mathbf{G}\,\neg p$ stating that p only holds finitely often, any disjunction of the form $(\mathbf{F}\,\mathbf{G}\,\neg p) \vee \psi_i$ is a tautology: either there are only finitely many occurrences of p, or otherwise, p holds infinitely often, and so p holds at least twice. Hence, we have, for both permutations, $\delta_\pi(\mathbf{F}\,\mathbf{G}\,\neg p) = \delta_{\pi'}(\mathbf{F}\,\mathbf{G}\,\neg p) = \delta\big((\mathbf{F}\,\mathbf{G}\,\neg p) \vee \bot\big) = \delta(\mathbf{F}\,\mathbf{G}\,\neg p)$.

The composition of a choice function δ with a permutation of a cleaving preserves rationality.

Lemma 38. *The composition δ_π of a choice function δ and a permutation $\pi : \mathbb{N} \to \mathcal{C}$ of a cleaving $\mathcal{C} \subseteq \mathcal{K}$ is a choice function.*

A composition generates a new choice function which in turn induces a rational contraction function that remains within the excerpt. Yet, the induced contraction function is not necessarily computable.

Theorem 39. *Let $\mathbb{E}$ accommodate rational contraction, and let $\mathcal{K} \in \mathbb{E}$. The following statements are equivalent:*

1. *The theory $\mathcal{K}$ is non-finitary.*
2. *There exists an* uncomputable *rational contraction function on $\mathcal{K}$ that remains in $\mathbb{E}$.*
3. *There exists an* uncomputable *fully rational contraction function on $\mathcal{K}$ that remains in $\mathbb{E}$.*

Proof Sketch. Let $\mathcal{K}$ be non-finitary, and δ the choice function of a (fully) rational contraction for $\mathcal{K}$ that remains in $\mathbb{E}$. Each permutation π of a cleaving $\mathcal{C} \subseteq \mathcal{K}$ induces a *distinct* (fully) rational contraction (with choice function δ_π) that remains in $\mathbb{E}$. At most countably many of these uncountably many (fully) rational contractions can be computable.

If $\mathcal{K}$ is finitary, every contraction function is computable, as it only has to consider finitely many formulae. ☐

Theorem 39 makes evident that uncomputability of AGM contraction is inevitable. In fact, there are uncountably many uncomputable contraction functions. The only way to avoid this uncomputability would be to restrain the expressiveness of the excerpt to the most trivial case: only finitary theories.

8. Effective Contraction in the Büchi Excerpt

Despite the strong negative result of Section 7, computability can still be harnessed in very particular excerpts: excerpts $\mathbb{E}$ in which for every theory, there exists at least one computable (fully) rational contraction function that remains in $\mathbb{E}$. We say that such an excerpt $\mathbb{E}$ *effectively accommodates* (fully) rational contraction. If belief contraction is to be computed for compendious logics, it is paramount to identify such excerpts as well as classes of computable contraction functions. In this section, we show that the Büchi excerpt of LTL effectively accommodates (fully) rational contraction.

For a contraction on a theory $\mathcal{K} \in \mathbb{E}_{\text{Büchi}}$ to remain in the Büchi excerpt, the underlying choice function must be designed such that the intersection of $\mathcal{K}$ with the selected CCTs corresponds to the support of a Büchi automaton. As CCTs and ultimately periodic traces are interchangeable (Lemma 20), and the support of a Büchi automaton is determined by the CCTs corresponding to its accepted ultimately periodic traces (Lemma 21), a solution is to design a selection mechanism, analogous to choice functions, that picks a single Büchi automaton instead of an (infinite) set of CCTs.

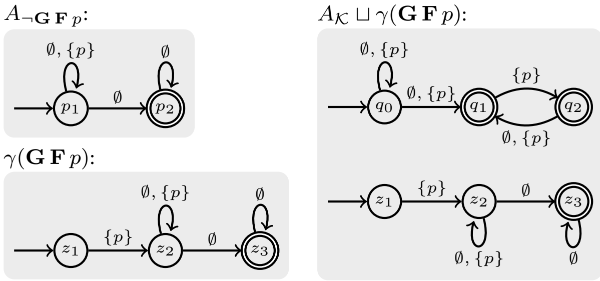


Figure 4: BCF contraction of $\mathbf{G}\,\mathbf{F}\,p$ from $\mathcal{S}(A_\mathcal{K})$ (Example 43).

Definition 40 (Büchi Choice Functions)**.** *A* Büchi choice function *γ maps each LTL formula to a single Büchi automaton, such that for all LTL formulae φ and ψ,*

(BF1) *the language accepted by $\gamma(\varphi)$ is non-empty;*

(BF2) *$\gamma(\varphi)$ supports $\neg\varphi$, if φ is not a tautology; and*

(BF3) *$\gamma(\varphi)$ and $\gamma(\psi)$ accept the same language, if $\varphi \equiv \psi$.*

Conditions **(BF1)** - **(BF3)** correspond to the respective conditions **(CF1)** - **(CF3)** in the definition of choice functions. Each Büchi choice function induces a contraction function.

Definition 41 (Büchi Contraction Functions)**.** *Let $\mathcal{K}$ be a theory in the Büchi excerpt and let γ be a Büchi choice function. The* Büchi Contraction Function (BCF) *on $\mathcal{K}$ induced by γ is the function*

$$\mathcal{K} \dot{-}_\gamma \varphi = \begin{cases} \mathcal{K} \cap \mathcal{S}(\gamma(\varphi)) & \textbf{\textit{if}}\ \varphi \notin Cn(\emptyset)\ \textit{and}\ \varphi \in \mathcal{K} \\ \mathcal{K} & \textbf{\textit{otherwise}} \end{cases}$$

All such contractions remain in the Büchi excerpt. Indeed, one can observe that if $\mathcal{K} = \mathcal{S}(A)$ for a Büchi automaton A, it holds that $\mathcal{K} \cap \mathcal{S}(\gamma(\varphi)) = \mathcal{S}(A \sqcup \gamma(\varphi))$. Recall from Section 2 that $\sqcup$ denotes the union of Büchi automata. The class of all rational contraction functions that remain in the Büchi excerpt corresponds exactly to the class of all BCFs.

Theorem 42. *A contraction function $\dot{-}$ on a theory $\mathcal{K} \in \mathbb{E}_{\text{Büchi}}$ is rational and remains within the Büchi excerpt if and only if $\dot{-}$ is a BCF.*

Example 43. *Let $\mathcal{K} = \mathcal{S}(A_\mathcal{K})$, for the Büchi automaton $A_\mathcal{K}$ shown in Fig. 3. To contract the formula $\mathbf{G}\,\mathbf{F}\,p$, a Büchi choice function γ may select the Büchi automaton $\gamma(\mathbf{G}\,\mathbf{F}\,p)$ shown in Fig. 4. This automaton supports $\neg\mathbf{G}\,\mathbf{F}\,p$; the automaton $A_{\neg \mathbf{G}\,\mathbf{F}\,p}$ is shown for reference. In fact, $\gamma(\mathbf{G}\,\mathbf{F}\,p)$ accepts precisely the traces satisfying $p \wedge \neg\mathbf{G}\,\mathbf{F}\,p$. The result of the contraction is the theory $\mathcal{S}(A_\mathcal{K}) \cap \mathcal{S}(\gamma(\mathbf{G}\,\mathbf{F}\,p))$, which corresponds to the theory $\mathcal{S}(A_\mathcal{K} \sqcup \gamma(\mathbf{G}\,\mathbf{F}\,p))$, whose supporting automaton is also shown in Fig. 4. The union $\sqcup$ is obtained by simply taking the union of states and transitions. This automaton does not support $\mathbf{G}\,\mathbf{F}\,p$, and therefore the contraction is successful.*

As BCFs capture all rational contractions within the excerpt, it follows from Theorem 39 that not all BCFs are computable. Note from Definition 41 that to achieve computability, it suffices to be able to: (i) decide if φ is a tautology, (ii) decide if $\varphi \in \mathcal{K}$, (iii) compute the underlying Büchi choice function γ, and (iv) compute the intersection of $\mathcal{K}$ with the support of $\gamma(\varphi)$. We already know that conditions (i) and (ii) are satisfied (Theorem 24). For condition (iv), recall that the intersection of the support of two automata is equivalent to the support of their union. As γ produces a Büchi automaton, and union of Büchi automata is computable, condition (iv) is also satisfied. Therefore, condition (iii) is the only one remaining. It turns out that (iii) is a necessary and sufficient condition to characterize all computable contraction functions within the Büchi excerpt.

Theorem 44. *Let $\dot{-}$ be a rational contraction function on a theory $\mathcal{K} \in \mathbb{E}_{\text{Büchi}}$, such that $\dot{-}$ remains in the Büchi excerpt. The operation $\dot{-}$ is computable iff $\dot{-} = \dot{-}_\gamma$ for some computable Büchi choice function γ .*

Note that there do indeed exist computable choice functions. As an example, the *full meet* contraction [1, 3] is computable. The corresponding Büchi choice function γ_{fm} is given by $\gamma_{\text{fm}}(\varphi) = A_{\neg\varphi}$ if φ is non-tautological, and $\gamma_{\text{fm}}(\varphi) = A_\top$ otherwise. This function is computable: the automata $A_{\neg\varphi}$ and $A_\top$ can be effectively constructed, and it is decidable whether the given LTL formula φ is a tautology.

As the *full meet* contraction is fully rational, we conclude that the Büchi excerpt effectively accommodates (fully) rational contraction.

9. Conclusion

We have investigated the computability of AGM contraction for the class of compendious logics, which embrace several logics used in computer science and AI. Due to the high expressive power of these logics, not all epistemic states admit a finite representation. Hence, the epistemic states that an agent can assume are confined to a space of theories, which depends on a method of finite representation. We have shown a severe negative result: no matter which form of finite representation we use, as long as it does not collapse to the finitary case, AGM contraction suffers from uncomputability. Precisely, there are uncountably many uncomputable (fully) rational contraction functions in all such expressive spaces. This negative result also impacts other forms of belief change. For instance, belief revision is interdefinable with belief contraction, via the Levi Identity. Therefore, it is likely that revision also suffers from uncomputability. Accordingly, uncomputability might span to iterated belief revision [41] , update and erasure [42], and pseudo-contraction [43], to cite a few. It is worth investigating uncomputability of these other operators.

In this work, we have focused on the AGM paradigm, and logics which are Boolean. We intend to expand our results for a wider class of logics by dispensing with the Boolean operators, and assuming only that the logic is AGM compliant. We believe the results shall hold in the more general case, as our negative results follow from cardinality arguments. On the other hand, several logics used in knowledge representation and reasoning are not AGM compliant, as for instance a variety of Description Logics [12]. In these logics, the *recovery* postulate ($\mathbf{K_5^-}$) can be replaced by the *relevance* postulate [44], and contraction functions can be properly defined. Such logics are called relevance-compliant. As relevance is an weakened version of recovery, the uncomputability results in this work translate to various relevance-compliant logics. However, it is unclear if all such logics are affected by uncomputability. We aim to investigate this issue in such logics.

Even if we have to coexist with uncomputability, we can still identify classes of operators which are guaranteed to be computable. To this end, we have introduced a novel class of contraction functions for LTL using Büchi automata, and identified the conditions needed for computability. This is an initial step towards the application of belief change in other areas, such as methods for automatically repairing systems [45]. The methods devised here for LTL form a foundation for the development of analogous strategies for other expressive logics, such as CTL, μ-calculus and many Description Logics. For example, in these logics, similarly to LTL, decision problems such as satisfiability and entailment have been solved using various kinds of automata, such as tree automata [46, 47].

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A. Proofs for Section 2 (Logics and Automata)

$\mathbb{L}$ is Boolean if for every $\varphi, \psi \in Fm$ there exist formulae $\neg\varphi$ resp. $\varphi \vee \psi$ such that:

($\neg_T$) $Cn(\{\varphi\}) \cap Cn(\{\neg\varphi\}) = Cn(\emptyset)$

($\neg_I$) $Cn(\{\varphi, \neg\varphi\}) = Fm$

($\vee_I$) If $\varphi \in Cn(X)$ then $(\varphi \vee \psi) \in Cn(X)$

($\vee_E$) If $\alpha \in Cn(X \cup \{\varphi\})$ and $\alpha \in Cn(X \cup \{\psi\})$, then $\alpha \in Cn(X \cup \{\varphi \vee \psi\})$

for all formulae α, φ, ψ and all sets of formulae X; and that $\mathcal{K} = \bigcap\{ \mathcal{K}' \in CCT_{\mathbb{L}} \mid \mathcal{K} \subseteq \mathcal{K}' \}$ for every theory $\mathcal{K}$.

Observation 5. *LTL is Tarskian and Boolean.*

Proof. It is easy to show that LTL is Tarskian. Regarding the Boolean operators, the disjunction is straightforward. The only interesting aspect is negation:

($\neg_T$) Let $\psi \in Cn(\varphi) \cap Cn(\neg\varphi)$, and let M be a Kripke structure. Assume $M \not\models \psi$. Then there exists an ultimately periodic trace π of M such that $\pi \not\models \psi$. But then the Kripke structure M_π with $Traces(M_\pi) = \{\pi\}$ either satisfies φ or $\neg\varphi$, and in either case it follows that $M_\pi \models \psi$. Thus we have a contradiction, and it must indeed be the case that every Kripke structure M satisfies ψ. Hence $\psi \in Cn(\emptyset)$.

($\neg_I$) Let $\varphi \in Fm_{LTL}$. Since there is no Kripke structure such that $M \models \varphi$ and $M \models \neg\varphi$, we can conclude that all such models satisfy ψ. Hence $\psi \in Cn(\{\varphi, \neg\varphi\})$.

□

Note that our notion of negation, in particular ($\neg_T$), is weaker than requiring $M \models \varphi$ or $M \models \neg\varphi$ for all formulae φ, a property not satisfied by LTL.

Proposition A.1 ([11])**.** *If $\mathcal{L}(A)$ is nonempty, then A accepts an ultimately periodic word.*

B. Proofs for Section 4 (Limits of Finite Representation)

Proposition B.1. *The theory $Cn(\mathbf{F}\,p)$ is not expressible via a finite set of models.*

Proof. Suppose there was a finite set of models, i.e., Kripke structures $\{M_1, \dots, M_n\}$ such that

$$\{ \varphi \in Fm_{LTL} \mid M_i \models \varphi, \text{for } i = 1, \dots, n \} = Cn(\mathbf{F}\,p)$$

Each Kripke structure M_i has some finite number of states m_i. Clearly, we must have $M_i \models \mathbf{F}\,p$. It follows that for every trace of M_i, p must hold at least once within the first m_i time steps: otherwise, there must be a cycle that can be reached and traversed without encountering an occurrence of p. If this were the case, there would also be an infinite trace corresponding to infinite repetition of this cycle, where p never holds; this would contradict $M_i \models \mathbf{F}\,p$.

Let now m be the maximum over all m_i for $i = 1, \dots, n$. Then each of the models M_i satisfies $\bigvee_{k=0}^{m} \mathbf{X}^k\,p$. Thus, $\bigvee_{k=0}^{m} \mathbf{X}^k\,p$ is in the theory represented by the models $M_1, \dots, M_n$. But this formula is not in $Cn(\mathbf{F}\,p)$, so we arrive at a contradiction. □

Lemma B.2. *A Tarskian, Boolean logic $\mathbb{L}$ is **(Discerning)** if and only if for every $\mathcal{K} \in CCT_{\mathbb{L}}$, there exists a formula φ with $\mathcal{K} = Cn(\varphi)$.*

Proof. Let $\mathbb{L}$ be a Tarskian, Boolean logic.

"⇒": Assume that $\mathbb{L}$ satisfies **(Discerning)**, i.e., that $\bigcap X = \bigcap Y$ implies $X = Y$ for all sets $X, Y \subseteq CCT_{\mathbb{L}}$. Let $\mathcal{K}$ be an arbitrary complete consistent theory.

Consider the set $X = CCT \setminus \{\mathcal{K}\}$. By assumption, since $X \neq CCT$, it follows that $\bigcap X \neq \bigcap CCT = Cn(\emptyset)$. Consequently, there must exist some formula $\alpha \in \bigcap X \setminus Cn(\emptyset)$.

Every CCT $\mathcal{K}' \in X$ contains α, and hence by consistency we have that $(\neg\alpha) \notin \mathcal{K}'$. However, since α is by assumption non-tautological, there must exist some CCT that does not contain it. The only choice is $\mathcal{K}$, and so we conclude that $\alpha \notin \mathcal{K}$ and by completeness, $(\neg\alpha) \in \mathcal{K}$. It follows that

$$Cn(\neg\alpha) = \bigcap\{ \hat{\mathcal{K}} \in CCT \mid Cn(\neg\alpha) \subseteq \hat{\mathcal{K}} \} = \bigcap\{\mathcal{K}\} = \mathcal{K}$$

Thus we have shown for an arbitrary CCT $\mathcal{K}$, that there exists a formula φ (namely, $\varphi :\equiv \neg\alpha$) such that $\mathcal{K} = Cn(\varphi)$.

"⇐": Assume that for every $\mathcal{K} \in CCT_{\mathbb{L}}$, there exists a formula φ such that $\mathcal{K} = Cn(\varphi)$. To show that $\mathbb{L}$ satisfies **(Discerning)**, we proceed by contraposition. To this end, let $X, Y \subseteq CCT_{\mathbb{L}}$ such that $X \neq Y$. Wlog. there exists some $\mathcal{K} \in X \setminus Y$. By assumption, there exists some formula $\varphi \in Fm_{\mathbb{L}}$ such that $\mathcal{K} = Cn(\varphi)$.

Let $\mathcal{K}'$ be any CCT other than $\mathcal{K}$. If it were the case that $\varphi \in \mathcal{K}'$, it would follow that $\mathcal{K} \subseteq \mathcal{K}'$. But this is a contradiction, as any strict superset of $\mathcal{K}$ must be inconsistent.

Thus for any CCT $\mathcal{K}'$ other than $\mathcal{K}$, it holds that $\varphi \notin \mathcal{K}'$ and hence $(\neg\varphi) \in \mathcal{K}'$. It follows that the formula $\neg\varphi$ is in the intersection $\bigcap Y$ (since $\mathcal{K} \notin Y$) but not in $\bigcap X$ (since $\mathcal{K} \in X$). We conclude that $\bigcap X \neq \bigcap Y$. □

Theorem 15. *The logics LTL, CTL, CTL*, μ-calculus and* monadic second-order logic *(MSO) are compendious.*

Proof. We refer to [11] for the definition of syntax and semantics of CTL, CTL* and the μ-calculus, and to [23] for MSO (there called SC). From these definitions, it is easy to see that these logics are Tarskian and Boolean. To show that they are non-finitary, it suffices to find an infinite set of pairwise non-equivalent formulae. For the case of LTL, such a set is for instance given for instance by $\{p, \mathbf{X}\,p, \mathbf{X}^2\,p, \dots\}$. It remains to show that the logics satisfy **(Discerning)**.

We begin by proving this for CTL. The same proof also applies directly to CTL* and μ-calculus (noting that CTL can be embedded in both these logics). Browne et al. [48] show that CTL formulae can characterize Kripke structures up to bisimilarity. More precisely, for every Kripke structure M, there exists a CTL formula φ_M such that φ_M is satisfied precisely by Kripke structures that are bisimilar to M. They also show that bisimilar Kripke structures in general satisfy the same CTL formulae. From these results, it follows that every CCT of CTL has a finite base, and thus by Lemma B.2, **(Discerning)** follows. We first show that every Kripke structure with a single initial state induces a CCT, and conversely, that every CCT is induced by a Kripke structure with a single state. The first part is trivial, as it follows directly from the semantics of negation (in CTL) that for every formula φ and every Kripke structure M with a single initial state, we have that $M \models \varphi$ or $M \models \neg\varphi$. Thus the set of formulae satisfied by M is a complete consistent theory. Let now $\mathcal{K}$ be a CCT. Since $\mathcal{K}$ is consistent, it is satisfied by some Kripke structure M. Wlog. we assume that M has only a single initial state: If not, we make all but one state non-initial; preserving satisfaction of $\mathcal{K}$. Then, by the result of Browne et al. [48], there exists a CTL formula φ_M characterizing M up to bisimilarity. Since $\mathcal{K}$ is complete, either $\varphi_M \in \mathcal{K}$ or $(\neg\varphi_M) \in \mathcal{K}$. But the latter would contradict $M \models \mathcal{K}$, hence we know that $\varphi_M \in \mathcal{K}$. It follows that $Cn(\varphi_M) \subseteq \mathcal{K}$, and since both are CCTs, this means $\mathcal{K} = Cn(\varphi_M)$.

It remains to prove **(Discerning)** for LTL and MSO. We also achieve this by showing that every CCT has a finite base. For LTL, this is shown in Lemmas 20 and C.2 below. The proof for MSO is analogous, noting that LTL can be embedded in MSO, and that MSO formulae can (like LTL formulae) be expressed as Büchi automata [23]. □

Theorem 16. *No encoding can represent every theory of a compendious logic.*

Proof. Since compendious logics are non-finitary, they have infinitely many theories. As the logic is Boolean and Tarskian, every theory can be described as a (possibly infinite) intersection of CCTs. Thus, there must be infinitely many CCTs. From **(Discerning)**, it follows that intersections of different sets of CCTs always yield different theories. As the powerset of the infinite set CCT is uncountable, we conclude that there exist uncountably many theories in the logic. However, an encoding can represent only countably many theories. □

C. Proofs for Section 5 (The Büchi Encoding of LTL)

As a basis for our results on LTL, we develop a tight connection between ultimately periodic traces and complete consistent formulae. We begin by defining formulae that uniquely identify an ultimately periodic trace.

Lemma C.1 (Identifying Formulae). *For every ultimately periodic trace π, there exists an LTL formula $id(\pi)$ that is satisfied by π and not by any other trace.*

Proof. Let $\pi = \rho\,\sigma^\omega$ be an ultimately periodic trace, where $\rho = a_1 \ldots a_n$ and $\sigma = b_0 \ldots b_m$. We define the formula

$$id(\pi) :\equiv \left(\bigwedge_{i=1}^{n} \mathbf{X}^{i-1}\, a_i \right) \wedge \left(\bigwedge_{i=0}^{m} \mathbf{X}^{n+i}\, b_i \right) \wedge \mathbf{X}^n\, \mathbf{G} \left(\bigwedge_{a\in\Sigma} a \to \mathbf{X}^{m+1}\, a \right)$$

where a letter $a \in \Sigma = \mathcal{P}(AP)$ abbreviates the formula $\bigwedge_{p\in a} p \wedge \bigwedge_{p\in AP\setminus a} \neg p$.

In this formula, the first conjunct establishes the (possibly empty) prefix $a_1 \ldots a_n$. The second conjunct establishes the subsequent (non-empty) sequence $b_0 \ldots b_m$. And finally, the third conjunct describes the shape of the trace, i.e. that after a prefix of length n it becomes periodic with a period of length $m + 1$. Any trace that satisfies these constraints is necessarily equal to π. □

In Section 5, we define the function $Th_{UP} : UP \to CCT_{LTL}$ with $Th_{UP}(\pi) := \{\, \varphi \in Fm_{LTL} \mid \pi \models \varphi \,\}$. This function can also be expressed via identifying formulae.

Lemma C.2. *It holds that $Th_{UP}(\pi) = Cn(id(\pi))$.*

Proof. Let $\varphi \in Th_{UP}(\pi)$. Then $\pi \models \varphi$. Any Kripke structure M that satisfies $id(\pi)$ must have $Traces(M) = \{\pi\}$, and hence M also satisfies φ. Therefore we conclude that $\varphi \in Cn(id(\pi))$.

Conversely, let $\varphi \in Cn(id(\pi))$. Consider a Kripke structure M_π with $Traces(M_\pi) = \{\pi\}$. As π is ultimately periodic, such a Kripke structure (with a finite number of states) exists. Then $M_\pi \models id(\pi)$, so by assumption also $M \models \varphi$. But this implies $\pi \models \varphi$ and thus $\varphi \in Th_{UP}(\pi)$. □

Lemma C.3. *For every $\pi \in UP$, $Th_{UP}(\pi)$ is a complete consistent theory. Hence, the function Th_{UP} is well-defined.*

Proof. Let $\pi = \rho\,\sigma^\omega$ be an ultimately periodic trace, where $\rho = a_1 \ldots a_n$ and $\sigma = b_1 \ldots b_n$. From Lemma C.2, it immediately follows that $Th_{UP}(\pi)$ is a theory.

To show consistency, we identify a model (i.e., a finite Kripke structure) that satisfies every formula in the theory. In particular, we construct a Kripke structure $M_\pi = (S, I, \to, \lambda)$ with as follows: The set of states is given by $S = \{q_1, \ldots, q_n, p_0, \ldots, p_m\}$ with initial states $I = \{q_0\}$. We define $\lambda(q_i) = a_i$ for $i \in \{1, \ldots, n\}$, and $\lambda(p_j) = b_j$ for $j \in \{0, \ldots, m\}$. Finally, $\to$ is the smallest relation with $q_i \to q_{i+1}$, $q_n \to p_0$, $p_j \to p_{j+1}$ and $p_m \to p_0$ for all $i \in \{1, \ldots, n-1\}$ and $j \in \{0, \ldots, m-1\}$.

This Kripke structure only has a single trace, namely $Traces(M_\pi) = \{\pi\}$. Thus it follows that $M_\pi \models id(\pi)$, and consequently $M_\pi \models Cn(id(\pi)) = Th_{UP}(\pi)$. Thereby we have shown that $Th_{UP}(\pi)$ is consistent.

It remains to show that $Th_{UP}(\pi)$ is complete, i.e., for every $\varphi \in Fm_{LTL}$, we must either have $\varphi \in Cn(id(\pi))$ or $(\neg\varphi) \in Cn(id(\pi))$. We distinguish two cases:

Case 1: $\pi \models \varphi$. Consider some Kripke structure M such that $M \models Cn(id(\pi))$. Then every trace of M must satisfy $id(\pi)$, i.e., it must hold that $Traces(M) = \{\pi\}$. Since $\pi \models \varphi$, it follows that $M \models \varphi$.

This reasoning applies to any M with $M \models Cn(id(\pi))$, and thus we have shown that $\varphi \in Cn(id(\pi))$.

Case 2: $\pi \models \neg\varphi$. We show that $(\neg\varphi) \in Cn(id(\pi))$, analogously to the previous case.

Since one of these two cases always applies, for any φ, we have shown that $Th_{UP}(\pi)$ is complete. □

Lemma C.4. *The function Th_{UP} is injective.*

Proof. Let $\pi_1, \pi_2 \in UP$ be ultimately periodic traces such that $Th_{UP}(\pi_1) = Th_{UP}(\pi_2)$. Since $\pi_2 \models id(\pi_2)$ it follows that $\pi_2 \models \varphi$ for any $\varphi \in Cn(id(\pi_2)) = Th_{UP}(\pi_2)$. But since $id(\pi_1) \in Th_{UP}(\pi_1)$, and the two theories are equal, this means that $\pi_2 \models id(\pi_1)$. By Lemma C.1, we conclude that $\pi_1 = \pi_2$. Thus the function Th_{UP} is injective. □

Lemma C.5. *The function Th_{UP} is surjective on CCT_{LTL}.*

Proof. Let $\mathcal{K}$ be a complete consistent theory. Since $\mathcal{K}$ is consistent, there exists a Kripke structure M such that $M \models \mathcal{K}$. Like any finite Kripke structure, M contains at least one ultimately periodic trace π. We will show that $\mathcal{K} = Th_{UP}(\pi)$, by considering each inclusion separately.

$\mathcal{K} \subseteq Th_{UP}(\pi)$**:** Let $\varphi \in \mathcal{K}$. Since $M \models \mathcal{K}$ and $\pi \in Traces(M)$, we know that $\pi \models \varphi$. It follows that $id(\pi) \models \varphi$, and hence $\varphi \in Cn(id(\pi)) = Th_{UP}(\pi)$.

$Th_{UP}(\pi) \subseteq \mathcal{K}$**:** Let $\varphi \in Th_{UP}(\pi)$. Then we know that $\pi \models \varphi$ and thus $\pi \not\models \neg\varphi$. It follows that also $M \not\models \neg\varphi$. Since $M \models \mathcal{K}$, this means that $(\neg\varphi) \notin \mathcal{K}$. But since $\mathcal{K}$ is complete, we conclude that $\varphi \in \mathcal{K}$.

Thereby we have shown that any CCT is equal to $Th_{UP}(\pi)$ for some ultimately periodic trace π, and thus the function Th_{UP} is surjective on CCT_{LTL}. □

Lemma 20. *The function Th_{UP} is a bijection.*

Proof. This follows from Lemmas C.4 and C.5. □

We have shown that in compendious logics, every CCT $\mathcal{K}$ has a finite base, i.e., a formula φ with $\mathcal{K} = Cn(\varphi)$. Lemmas 20 and C.2 give us a concrete idea of these finite bases for the case of LTL: every CCT of LTL is equal to $Cn(id(\pi))$, for some ultimately periodic trace π.

Next, we make use of this connection between CCTs and ultimately periodic traces to characterize the support of a Büchi automaton.

Lemma 21. *The support of a Büchi automaton A satisfies*

$$\mathcal{S}(A) = \bigcap \{\, Th_{UP}(\pi) \mid \pi \in \mathcal{L}(A) \cap UP \,\}\,.$$

Proof. Let $\varphi \in \mathcal{S}(A)$. Then $\pi \models \varphi$ for each $\pi \in \mathcal{L}(A)$, and in particular, for each $\pi \in \mathcal{L}(A) \cap UP$. Thus, $\varphi \in Th_{UP}(\pi)$ for each such ultimately periodic π, and hence $\varphi \in \bigcap\{\, Th_{UP}(\pi) \mid \pi \in \mathcal{L}(A) \cap UP \,\}$.

For the converse inclusion, let $\varphi \in \bigcap\{\, Th_{UP}(\pi) \mid \pi \in \mathcal{L}(A) \cap UP \,\}$. Then $\pi \models \varphi$ for each ultimately periodic trace in $\mathcal{L}(A)$. Suppose there was a trace π' that was not ultimately periodic, such that $\pi' \not\models \varphi$. Then the set $\mathcal{L}(A) \setminus \mathcal{L}(A_\varphi)$ would be non-empty. As the difference of two languages recognized by Büchi automata can again be recognized by a Büchi automaton, and any Büchi automaton that recognizes a nonempty language accepts at least one ultimately periodic trace, we conclude that there exists an ultimately periodic trace in $\mathcal{L}(A)$ that does not satisfy φ. This is however a contraction. Hence, our assumption was incorrect and indeed we have $\pi' \models \varphi$ for all $\pi' \in \mathcal{L}(A)$. We conclude that $\varphi \in \mathcal{S}(A)$.

We have shown both inclusions, so the equality holds. □

Theorem 22. *The support of a Büchi automaton is a theory.*

Proof. This is a direct consequence of Lemma 21, as the intersection of (complete consistent) theories is a theory. □

Lemma C.6. *Let A be a Büchi automaton, and φ and LTL formula. Then it holds that $\mathcal{S}(A) + \varphi = \mathcal{S}(A \sqcap A_\varphi)$.*

Proof. Let $\psi \in \mathcal{S}(A) + \varphi = Cn(\mathcal{S}(A) \cup \{\varphi\})$. By Lemma 21, it suffices to show that each ultimately periodic trace $\pi \in \mathcal{L}(A \sqcap A_\varphi)$ satisfies ψ. To see this, consider a Kripke structure M_π with $Traces(M_\pi) = \{\pi\}$. Clearly, $\pi \models \mathcal{S}(A) \cup \{\varphi\}$, and so $M_\pi \models \mathcal{S}(A) \cup \{\varphi\}$. This implies that $M_\pi \models \psi$, and hence $\pi \models \psi$. As this holds for all ultimately periodic traces $\pi \in \mathcal{L}(A \sqcap A_\varphi)$, we conclude that $\psi \in \mathcal{S}(A \sqcap A_\varphi)$.

For the converse inclusion, let $\psi \in \mathcal{S}(A \sqcap A_\varphi)$. We have to show that any Kripke structure M with $M \models \mathcal{S}(A) \cup \{\varphi\}$ also satisfies ψ. Suppose this was not the case, i.e., that $M \not\models \psi$. Then there exists an ultimately periodic trace $\pi \in \mathcal{L}(A_M) \setminus \mathcal{L}(A_\psi)$. As $\pi \notin \mathcal{L}(A)$, we have that every trace in $\mathcal{L}(A)$ satisfies $\neg id(\pi)$, and hence $(\neg id(\pi)) \in \mathcal{S}(A)$. But this contradicts the fact that $M \models \mathcal{S}(A)$. Hence the supposition was wrong, and we have indeed that $M \models \psi$.

We have shown both inclusions, so the equality holds. □

D. Proofs for Section 6 (AGM Accommodation)

Proposition D.1. *An excerpt is open-minded if it contains any of the following:*

- *the inconsistent theory,*
- *all theories with finite bases, or*
- *all theories induced by a finite set of models.*

Proof. The inconsistent theory contains all formulae; and in particular it contains both φ and $\neg\varphi$ for every formula φ.

An excerpt that contains all theories with finite bases in particular contains both the theory $Cn(\varphi)$ and the theory $Cn(\neg\varphi)$.

In a compendious logic, there must be some (in fact, infinitely many) formulae that are neither tautological nor inconsistent; otherwise the logic would be finitary. Let us pick one such φ. Then because φ is consistent, there exists a model M such that $M \models \varphi$, and the theory $\mathcal{K} = \{\psi \mid M \models \psi\}$ is in the excerpt (because it corresponds to the singleton set of models $\{M\}$). Because φ is not tautological, there also exists a model M' such that $M' \models \neg\varphi$, and by similar reasoning, $\neg\varphi$ thus also appears in the excerpt. □

Theorem 26 (Impossibility of Closedness)**.** *If $\mathbb{E}$ is an open-minded, finitely representable excerpt of a compendious logic, then $\mathbb{E}$ is not closed under rational contraction.*

Proof. Let $\mathbb{E}$ be an open-minded excerpt. By definition, there is a formula φ and theories $\mathcal{K}, \mathcal{K}' \in \mathbb{E}$ such that $\varphi \in \mathcal{K}$ and $\neg\varphi \in \mathcal{K}'$. As compendious logics are non-finitary and closed under classical negation, there are infinitely many CCTs (because otherwise they would only have finitely many theories, but either there are infinitely many uncomparable formulae: in which case their closures form infinitely many uncomparable theories: of there is an infinite chain of implications betwen the formulae: and thus their closures also form an infinite chain of distinct theories). Since every CCT is either a complement of φ or of $\neg\varphi$, we have that either $\overline{\omega}(\varphi)$ or $\overline{\omega}(\neg\varphi)$ presents infinitely many CCTs. Without loss of generality, let $\overline{\omega}(\varphi)$ have infinitely many CCTs.

Let Γ be a set of choice functions such that, $\delta \in \Gamma$ iff for all formulae ψ:

1. $\delta(\psi) \in (\mathcal{P}(\overline{\omega}(\psi)) \setminus \{\emptyset\})$, if $\varphi \equiv \psi$;
2. $\delta(\psi) = CCT$, if $\psi \equiv \top$;
3. $\delta(\psi) = \overline{\omega}(\psi)$, otherwise.

Intuitively, each choice function $\delta \in \Gamma$ behaves by picking all complements of a formula ψ to be contracted, as long as ψ is not equivalent to φ (condition 3). As tautologies have no complements, δ chooses all CCTs (condition 2). In the case that $\psi \equiv \varphi$, then δ choses some complements of φ. All choice functions in Γ differ only at this condition. In fact, each subset of the complements of φ gives a different choice function in Γ. Therefore, Γ has the same cardinality as $\mathcal{P}(\overline{\omega}(\varphi))$. Thus, as φ has infinitely many complements, we get that $\mathcal{P}(\overline{\omega}(\varphi))$ is uncountable. This means that Γ is also uncountable.

Due to **(Discerning)**, we have that each choice function in Γ yields a different contraction for φ, which means that there are uncountably many rational outcomes to contract φ from $\mathcal{K}$. However, $\mathbb{E}$ is countable, as it is finitely representable. Therefore, $\mathbb{E}$ is not closed under rational contraction. □

Proposition 27. *An excerpt is closed under fully rational contraction iff it is closed under rational contraction.*

Proof. One direction is trivial: Any fully rational contraction is a rational contraction, thus an excerpt closed under rational contraction is also closed under fully rational contraction.

For the other direction: Let an excerpt be closed under fully rational contraction, let $\mathcal{K}$ be a theory of the excerpt, and let φ be a formula. Let $\dot{-}$ be any rational contraction, and $\mathcal{K}' = \mathcal{K} \dot{-} \varphi$. We have to show that $\mathcal{K}'$ is in the excerpt. We distinguish two cases:

Case 1: $\varphi \notin \mathcal{K}$. Then $\mathcal{K}' = \mathcal{K}$ is in the excerpt, by assumption.

Case 2: $\varphi \in \mathcal{K}$. Then $\mathcal{K}' = \mathcal{K} \cap \bigcap \delta(\varphi)$, where δ is the choice function underlying $\dot{-}$.

Let $<$ be the preference relation such that $S_1 < S_2$ iff $S_1 \notin \delta(\varphi)$ and $S_2 \in \delta(\varphi)$. This relation trivially satisfies mirroring and maximal cut. For the choice function $\delta_<$, it holds that $\delta_<(\varphi) = \delta(\varphi)$, and hence the fully rational contraction assigns $\mathcal{K} \dot{-}_< \varphi = \mathcal{K}'$. By assumption of closedness under fully rational contraction, we now have that $\mathcal{K}'$ is in the excerpt.

□

Definition D.2. *A theory is $\mathcal{K}$ is supreme iff $\mathcal{K}$ is not tautological and for all $\alpha \in \mathcal{K}$, either $Cn(\alpha) = \mathcal{K}$ or $Cn(\alpha) = Cn(\emptyset)$.*

Observe that by definition, supreme theories always have a finite base.

Let r be a function that ranks each CCT to a negative integer, such that distinct CCTs are ranked to different negative integers. Let $<_r$ be the induced relation from r that is $X <_r Y$ iff $r(X) < (Y)$. Note that $<_r$ is a strict total order. Also note that it satisfies **(Maximal Cut)** and due to totality it satisfies **(Mirroring)**.

Given a theory $\mathcal{K}$, we define

$$\mathcal{K} \circ \varphi = \begin{cases} \mathcal{K} \cap \bigcap \min_{<_r}(\overline{\omega}(\varphi)) & \text{if } \varphi \not\equiv \top \text{ and } \varphi \in \mathcal{K} \\ \mathcal{K} & \text{otherwise} \end{cases}$$

Observation D.3. *$\mathcal{K} \circ \varphi$ is fully AGM rational.*

Proof. Observe that by definition, $\circ$ is a blade contraction function and therefore it is fully AGM rational. □

It remains to show that $\circ$ remains within the excerpt.

Proposition D.4. *If $\mathbb{E}$ accommodates contraction and $\mathcal{K} \in \mathbb{E}$, then for all formula φ, $\mathcal{K} \circ \varphi \in \mathbb{E}$*

Proof. If $\varphi \in Cn(\emptyset)$ or $\varphi \notin \mathcal{K}$, then by definition $\mathcal{K} \circ \varphi = \mathcal{K}$, and by hypothesis, $\mathcal{K}$ is within the excerpt. The proof proceeds for the case that $\varphi \not\equiv \top$ and $\varphi \in \mathcal{K}$. Let $\mathcal{K}' = \mathcal{K} \circ \varphi$. Thus, from the definition of $\circ$, we have $\mathcal{K}' = \mathcal{K} \cap \bigcap \min_{<_\mathcal{K}}(\overline{\omega}(\varphi))$. As $<_r$ is strictly total, $\min_{<_r}(\overline{\omega}(\varphi))$ is a singleton set $\{M\}$, which implies that $\mathcal{K}' = \mathcal{K} \cap M$.

Let $X = CCT \setminus \{M\}$. Thus, $\bigcap X$ is a supreme theory, which means that there is some formula α such that $\bigcap X = Cn(\alpha)$. Therefore, as M is the only counter CCT of α, we get that the only solution to contract α is $\mathcal{K} \cap M$. By hypothesis, the excerpt $\mathbb{E}$ accommodates contraction. Thus, there is some contraction operator $\dot{-}$ on $\mathcal{K}$ such that $\text{img}(\dot{-}) \subseteq \mathbb{E}$. Therefore, $\mathcal{K} \dot{-} \alpha = \mathcal{K} \cap M$. Thus, as by hypothesis $\dot{-}$ remains within the excerpt, we have that $\mathcal{K} \dot{-} \alpha \in \mathbb{E}$ which implies that $\mathcal{K}' \in \mathbb{E}$. Therefore, as $\mathcal{K}' = \mathcal{K} \circ \varphi$, we have that $\mathcal{K} \circ \varphi \in \mathbb{E}$. □

Proposition 29. *An excerpt $\mathbb{E}$ accommodates rational contraction iff $\mathbb{E}$ accommodates fully rational contraction.*

Proof. The fact that accommodation of fully rational contraction implies accommodation of rational contraction is straightforward. The opposite direction follows from Observation D.3 and Proposition D.4. □

E. Proofs for Section 7 (Uncomputability of Contraction)

Let us see an example of finitary theories in LTL.

Example E.1. *For an ultimately periodic trace $\pi \in UP$, consider the theory $Cn(\neg id(\pi))$. This theory contains only 2 equivalence classes, namely the equivalence class of $\neg id(\pi)$ and the tautologies. Specifically, consider some φ with $\neg id(\pi) \models \varphi$. This means that for all $\pi' \in UP \setminus \{\pi\}$, we have that $\pi' \models \varphi$. Now either $\pi \models \varphi$, and hence φ is a tautology; or $\pi \not\models \varphi$, so*

$$\{\pi' \in UP \mid \pi' \models \varphi\} \subseteq UP \setminus \{\pi\} = \{\pi' \in UP \mid \pi' \models \neg id(\pi)\}$$

which means $\varphi \models \neg id(\pi)$, and hence (with the assumption above) $\varphi \equiv \neg id(\pi)$.

The formulae $\neg id(\pi)$ are the weakest non-tautological formulae of LTL; their negation $id(\pi)$ are the bases of CCTs (i.e., they are the strongest consistent formulae of LTL). In general, a finitary belief state must be very weak; it can only imply finitely many formulae. Note that the implied formulae can be weakened further by disjoining them with arbitrary other formulae; and still such weakening only results in finitely many different beliefs. In other words: A finitary theory is only finitely many beliefs away from the tautological theory; and those finitely many beliefs must be very weak, or they would imply infinitely many consequences.

E.1. Existence of Infinite Cleavings

We prove that every non-finitary theory must contain an infinite cleaving.

Definition E.2. *The decomposition of a theory in terms of CCTs is given by the function*

$$decomp(\mathcal{K}) = \{X \in CCT \mid \mathcal{K} \subseteq X\}.$$

Lemma E.3. *A theory is non-finitary iff $CCT \setminus decomp(\mathcal{K})$ is infinite.*

Proof. We show the two implications separately.

"⇒": Contrapositively, assume that $CCT \setminus decomp(\mathcal{K})$ is finite, in particular let $CCT \setminus decomp(\mathcal{K}) = \{C_1, \ldots, C_n\}$. We know that in a logic with the above assumptions, every CCT has a finite base. In particular, let $C_i = Cn(\varphi_i)$ for $i = 1, \ldots, n$. Then every formula in $\mathcal{K}$ is equivalent to $\bigwedge_{C_i \in X} \neg\varphi_i$ for some $X \subseteq \{C_1, \ldots, C_n\}$.

To see this, take some $\alpha \in \mathcal{K}$. Let $X = \{C_i \mid \alpha \notin C_i\}$. Then $\varphi_i \models \neg\alpha$ for every $C_i \in X$, therefore $\alpha \models \neg\varphi_i$, and thus $\alpha \models \bigwedge_{C_i \in X} \neg\varphi_i$. For the reverse entailment, note that every CCT not in X is either one of the remaining C_i or in $decomp(\mathcal{K})$, and thus every such CCT contains α. Therefore, any CCT containing the formula $\bigwedge_{C_i \in X} \neg\varphi_i$ must also contain α. In other words, $\bigwedge_{C_i \in X} \neg\varphi_i \models \alpha$.

Since every equivalence class of formulae in $\mathcal{K}$ corresponds to one of the 2^n possible choices of X, we conclude that $\mathcal{K}$ is finitary.

"⇐": Suppose $CCT \setminus decomp(\mathcal{K})$ is infinite, and let $CCT \setminus decomp(\mathcal{K}) = \{C_1, C_2, \ldots\}$ be a duplicate-free enumeration of the set. For every C_i, we know that there exists a finite base φ_i. Consequently, the infinitely many formulae $\neg\varphi_i$ are all in $\mathcal{K}$. These formulae are pairwise non-equivalent (otherwise we would have $C_i = C_j$). Thus we conclude that $\mathcal{K}$ is non-finitary.

□

Lemma 35. *Every non-finitary theory contains a cleaving.*

Proof. Let $\mathcal{K}$ be a non-finitary theory. By the above lemma, we know that $CCT \setminus decomp(\mathcal{K})$ is infinite. Let $CCT \setminus decomp(\mathcal{K}) = \{C_1, C_2, \ldots\}$ be a duplicate-free enumeration of the set, and let $C_i = Cn(\varphi_i)$ for each i. We consider the set of formulae $\{\neg\varphi_i \mid C_i \in CCT \setminus decomp(\mathcal{K})\}$. This set is infinite, and the formulae are pairwise non-equivalent. For every pair C_i, C_j with $i \neq j$, the formula $(\neg\varphi_i) \vee (\neg\varphi_j)$ is a tautology. □

Lemma 38. *The composition δ_π of a choice function δ and a permutation $\pi : \mathbb{N} \to \mathcal{C}$ of a cleaving $\mathcal{C} \subseteq \mathcal{K}$ is a choice function.*

Proof. We show that δ_π satisfies all three conditions of choice functions, for all formulae φ, ψ:

To show: $\delta_\pi(\varphi) \neq \emptyset$. Since $\delta_\pi(\varphi) = \delta(\varphi \vee \min_\pi(\varphi))$, and by assumption that δ is a choice function, we have $\delta(\varphi \vee \min_\pi(\varphi)) \neq \emptyset$, the result follows.

To show: If $\varphi \notin Cn(\emptyset)$, then $\delta_\pi(\varphi) \subseteq \overline{\omega}(\varphi)$. Suppose that $\varphi \notin Cn(\emptyset)$. We have either $\min_\pi(\varphi) = \pi(i)$ for some i, or $\min_\pi(\varphi) = \bot$. In the latter case, $\varphi \vee \min_\pi(\varphi) \equiv \varphi$, and thus by assumption that δ is a choice function, $\delta(\varphi \vee \min_\pi(\varphi)) = \delta(\varphi) \subseteq \overline{\omega}(\varphi)$.

Let us thus now assume that $\min_\pi(\varphi) = \pi(i)$ for some i. Then $\overline{\omega}(\varphi \vee \pi(i)) = \overline{\omega}(\varphi) \cap \overline{\omega}(\pi(i)) \neq \emptyset$, so $\varphi \vee \pi(i)$ is not a tautology. Since δ is a choice function, we conclude that $\delta_\pi(\varphi) = \delta(\varphi \vee \pi(i)) \subseteq \overline{\omega}(\varphi \vee \pi(i)) \subseteq \overline{\omega}(\varphi)$.

To show: If $\varphi \equiv \psi$, then $\delta_\pi(\varphi) = \delta_\pi(\psi)$. Suppose $\varphi \equiv \psi$, then we have $\overline{\omega}(\varphi) = \overline{\omega}(\psi)$, and thus $\min_\pi(\varphi) = \min_\pi(\psi)$. It follows that $\varphi \vee \min_\pi(\varphi) \equiv \psi \vee \min_\pi(\psi)$. Since δ is a choice function, we conclude that $\delta_\pi(\varphi) = \delta(\varphi \vee \min_\pi(\varphi)) = \delta(\psi \vee \min_\pi(\psi)) = \delta_\pi(\psi)$.

Thus we have shown that δ_π is a choice function. □

E.2. Uncomputability

We prove the main result of this section: A non-finitary theory that admits any contraction must admit uncomputable contractions.

Observation E.4. *It is easy to see that* $\mathrm{img}(\delta_\pi) \subseteq \mathrm{img}(\delta)$.

Lemma E.5. *Let $\pi, \pi' : \mathbb{N} \to \mathcal{C}$ be two* distinct *permutations of $\mathcal{C}$. Then there exists a formula $\alpha \in \mathcal{K}$ such that $\delta_\pi(\alpha) \neq \delta_{\pi'}(\alpha)$.*

Proof. Since π and π' are different permutations, there must exist some indices $i, j, i', j' \in \mathbb{N}$ with $i < j$ and $i' < j'$ such that $\pi'(i') = \pi(j)$ and $\pi'(j') = \pi(i)$.

Consider now the formula $\alpha := \pi(i) \wedge \pi(j)$. Since $\pi(i), \pi(j)$ are in $\mathcal{K}$, we clearly have $\alpha \in \mathcal{K}$.

As the next step, we show that $\min_\pi(\alpha) = \pi(i)$:

- Note that $\overline{\omega}(\alpha) = \overline{\omega}(\pi(i)) \cup \overline{\omega}(\pi(j))$.
- Since $\mathcal{C}$ does not contain a tautology, $\overline{\omega}(\pi(i))$ is non-empty, and hence $\overline{\omega}(\alpha) \cap \overline{\omega}(\pi(i)) \neq \emptyset$.
- Furthermore, the complements of $\pi(i)$ and any $\pi(k)$ with $k \neq i$ are disjoint (by property **(CL2)** of cleavings), and the same holds for $\pi(j)$.
- Hence, the only k such that $\overline{\omega}(\alpha) \cap \overline{\omega}(k) \neq \emptyset$ are $k = i$ and $k = j$.
- Finally, recall that $i < j$.

It follows that $\min_\pi(\alpha) = \pi(i)$. Consequently, $\alpha \vee \min_\pi(\alpha) \equiv (\pi(i) \wedge \pi(j)) \vee \pi(i) \equiv \pi(i)$, and thus it follows that $\delta_\pi(\alpha) = \delta(\pi(i)) \subseteq \overline{\omega}(\pi(i))$. Noting that $\alpha = \pi'(j') \wedge \pi'(i')$, and applying analogous reasoning, we have that $\delta_{\pi'}(\alpha) \subseteq \overline{\omega}(\pi'(i')) = \overline{\omega}(\pi(j))$.

Thus we have shown that $\delta_\pi(\alpha) \subseteq \overline{\omega}(\pi(i))$ and $\delta_{\pi'}(\alpha) \subseteq \overline{\omega}(\pi(j))$. With the disjointness of complements in a cleaving **(CL2)**, it follows that $\delta_\pi(\alpha) \cap \delta_{\pi'}(\alpha) = \emptyset$. But since $\delta_\pi, \delta_{\pi'}$ are choice functions, and thus $\delta_\pi(\alpha)$ and $\delta_{\pi'}(\alpha)$ cannot be empty, we conclude that $\delta_\pi(\alpha) \neq \delta_{\pi'}(\alpha)$. □

Lemma E.6. *Let $\pi, \pi' : \mathbb{N} \to \mathcal{C}$ be two* distinct *permutations of $\mathcal{C}$. Then the induced contractions differ, i.e., it holds that $\dot{-}_{\delta_\pi} \neq \dot{-}_{\delta_{\pi'}}$.*

Proof. By Lemma E.5, there exists a formula α such that $\delta_\pi(\alpha) \neq \delta_{\pi'}(\alpha)$. Consider now the following sets of CCTs: $decomp(\mathcal{K}) \cup \delta_\pi(\alpha)$ and $decomp(\mathcal{K}) \cup \delta_{\pi'}(\alpha)$. Since $decomp(\mathcal{K})$ contains only CCTs that contain α, whereas $\delta_\pi(\alpha)$ and $\delta_{\pi'}(\alpha)$ contain only CCTs that do not contain α, each of the two unions has no overlap. Therefore, we conclude that $decomp(\mathcal{K}) \cup \delta_\pi(\alpha) \neq decomp(\mathcal{K}) \cup \delta_{\pi'}(\alpha)$. With **(Compendious)**, it follows that

$$\mathcal{K} \dot{-}_{\delta_\pi} \alpha = \mathcal{K} \cap \delta_\pi(\alpha) = \bigcap(decomp(\mathcal{K}) \cup \delta_\pi(\alpha))$$
$$\bigcap(decomp(\mathcal{K}) \cup \delta_\pi(\alpha)) \neq \bigcap(decomp(\mathcal{K}) \cup \delta_{\pi'}(\alpha))$$
$$\bigcap(decomp(\mathcal{K}) \cup \delta_{\pi'}(\alpha)) = \mathcal{K} \cap \delta_{\pi'}(\alpha) = \mathcal{K} \dot{-}_{\delta_{\pi'}} \alpha$$

Since $\dot{-}_{\delta_\pi}$ and $\dot{-}_{\delta_{\pi'}}$ differ on α, they must be different contractions. ☐

Lemma E.7. *Let $\dot{-}$ be a rational contraction on a non-finitary theory $\mathcal{K}$, such that $\dot{-}$ remains in the excerpt $\mathbb{E}$. Then there exist uncountably many rational contractions on $\mathcal{K}$ that remain in $\mathbb{E}$.*

Proof. We have shown that $\mathcal{K}$ contains an infinite cleaving (Lemma 35). By Lemmas 38 and E.6, each permutation of this infinite cleaving induces a distinct rational contraction; and by Observation E.4, each of these contractions remains in $\mathbb{E}$. Since there are uncountably many permutations of an infinite set, the result follows. ☐

Lemma E.8. *[18] An ECF is fully rational iff its choice function satisfies both conditions:*

(C1) *$\delta(\varphi \wedge \psi) \subseteq \delta(\varphi) \cup \delta(\psi)$, for all formulae φ and ψ;*

(C2) *For all formulae φ and ψ, if $\overline{\omega}(\varphi) \cap \delta(\varphi \wedge \psi) \neq \emptyset$ then $\delta(\varphi) \subseteq \delta(\varphi \wedge \psi)$*

Lemma E.9. *For every permutation π, if δ satisfies **(C1)**, then so does δ_π.*

Proof. Let φ and ψ be two formulae. We distinguish three cases:

Case 1: $\min_\pi(\varphi) = \min_\pi(\psi) = \bot$**.** In this case, $\overline{\omega}(\varphi)$ and $\overline{\omega}(\psi)$ must both be disjoint from $\overline{\omega}(\pi(k))$ for all k. It follows that the same holds for $\overline{\omega}(\varphi \wedge \psi) = \overline{\omega}(\varphi) \cup \overline{\omega}(\psi)$, and hence also $\min_\pi(\varphi \wedge \psi) = \bot$. We conclude:

$$\delta_\pi(\varphi\wedge\psi) = \delta(\varphi\wedge\psi) \subseteq \delta(\varphi)\cup\delta(\psi) = \delta_\pi(\varphi)\cup\delta_\pi(\psi)$$

Case 2: $\min_\pi(\varphi) = \pi(i), \min_\pi(\psi) = \pi(j)$ **for some** i, j**.** In this case, let us assume wlog. that $i \leq j$. It follows that both $\overline{\omega}(\varphi)$ and $\overline{\omega}(\psi)$ are disjoint from $\overline{\omega}(\pi(k))$ for all $k < i$. Consequently, $\overline{\omega}(\varphi \wedge \psi) = \overline{\omega}(\varphi) \cup \overline{\omega}(\psi)$ is also disjoint from $\overline{\omega}(\pi(k))$ for all $k < i$; but is not disjoint from $\overline{\omega}(\pi(i))$. Hence, we have $\min_\pi(\varphi \wedge \psi) = \pi(i)$.

It holds that $(\varphi\wedge\psi)\vee\pi(i) \equiv (\varphi\vee\pi(i))\wedge(\psi\vee\pi(i))$, and hence we conclude that

$$\delta_\pi(\varphi\wedge\psi) = \delta((\varphi\wedge\psi)\vee\pi(i)) = \delta((\varphi\vee\pi(i))\wedge(\psi\vee\pi(i)))$$

If we now have $i = j$, then it follows (as δ satisfies **(C1)**) that

$$\delta_\pi(\varphi\wedge\psi) = \delta(\varphi\vee\pi(i))\cup\delta(\psi\vee\pi(i)) = \delta_\pi(\varphi)\cup\delta_\pi(\psi)$$

and we are done. If $i < j$, then we must have $\overline{\omega}(\psi) \cap \overline{\omega}(\pi(i)) = \emptyset$, and $\psi \vee \pi(i)$ is a tautology. Hence,

$$\delta_\pi(\varphi\wedge\psi) = \delta(\varphi\vee\pi(i)) = \delta_\pi(\varphi) \subseteq \delta_\pi(\varphi)\cup\delta_\pi(\psi)$$

Case 3: $\{\min_\pi(\varphi), \min_\pi(\psi)\} = \{\pi(i), \bot\}$ **for some** i**.** In this case, let us assume wlog. that $\min_\pi(\varphi) = \bot$ and $\min_\pi(\psi) = \pi(i)$. Then $\overline{\omega}(\varphi)$ is disjoint from $\overline{\omega}(\pi(k))$ for all k. Since $\overline{\omega}(\varphi \wedge \psi) = \overline{\omega}(\varphi) \cup \overline{\omega}(\psi)$, it follows that $\min_\pi(\varphi \wedge \psi) = \pi(i)$.

We know $\overline{\omega}(\varphi)$ is disjoint from $\overline{\omega}(\pi(i))$, and thus $\varphi \vee \pi(i)$ is a tautology. It follows that $(\varphi \wedge \psi) \vee \pi(i) \equiv \psi \vee \pi(i)$. Hence,

$$\delta_\pi(\varphi\wedge\psi) = \delta(\psi\vee\pi(i)) = \delta_\pi(\psi) \subseteq \delta_\pi(\varphi)\cup\delta_\pi(\psi)$$

Thus we have shown that δ_π indeed satisfies **(C1)**. ☐

Lemma E.10. *For every permutation π, if δ satisfies **(C2)**, then so does δ_π.*

Proof. Let φ and ψ be formulae, and assume that $\overline{\omega}(\varphi) \cap \delta_\pi(\varphi \wedge \psi) \neq \emptyset$. We distinguish four cases:

Case 1 : $\min_\pi(\varphi) = \min_\pi(\psi) = \bot$. In this case, $\overline{\omega}(\varphi)$ and $\overline{\omega}(\psi)$ must both be disjoint from $\overline{\omega}(\pi(k))$ for all k. It follows that the same holds for $\overline{\omega}(\varphi \wedge \psi) = \overline{\omega}(\varphi) \cup \overline{\omega}(\psi)$, and hence also $\min_\pi(\varphi \wedge \psi) = \bot$. Then $\delta_\pi(\varphi) = \delta(\varphi)$, $\delta_\pi(\psi) = \delta(\pi)$ and $\delta_\pi(\varphi \wedge \psi) = \delta(\varphi \wedge \psi)$. Since δ satisfies **(C2)**, the result follows.

Case 2 : $\min_\pi(\varphi) = \pi(i)$, and either $\min_\pi(\psi) = \bot$ or $\min_\pi(\psi) = \pi(j)$ for some $j \geq i$. In this case, we observe that $\min_\pi(\varphi \wedge \psi) = \pi(i)$.

Since φ and $\pi(i)$ have shared complements, $(\varphi \wedge \psi)\vee\pi(i)$ cannot be a tautology. Then, by hypothesis and the fact that δ is a choice function, we have

$$\begin{aligned}\emptyset \neq \overline{\omega}(\varphi) \cap \delta_\pi(\varphi \wedge \psi) &= \overline{\omega}(\varphi) \cap \delta((\varphi \wedge \psi) \vee \pi(i))\\ &\subseteq \overline{\omega}(\varphi) \cap \overline{\omega}((\varphi \wedge \psi) \vee \pi(i))\\ &\subseteq \overline{\omega}(\varphi \vee \pi(i))\end{aligned}$$

Then we have

$$\begin{aligned}&\delta((\varphi \vee \pi(i)) \wedge (\psi \vee \pi(i))) \cap \overline{\omega}(\varphi \vee \pi(i))\\ &= \delta((\varphi \wedge \psi) \vee \pi(i)) \cap \overline{\omega}(\varphi \vee \pi(i)) \neq \emptyset.\end{aligned}$$

and by **(C2)** for δ, we conclude that

$$\delta_\pi(\varphi) = \delta(\varphi\vee\pi(i)) \subseteq \delta((\varphi\vee\pi(i))\wedge(\psi\vee\pi(i))) = \delta_\pi(\varphi\wedge\psi)$$

Case 3 : $\min_\pi(\psi) = \pi(j)$ for some j, and either $\min_\pi(\varphi) = \pi(i)$ for $i > j$ or $\min_\pi(\varphi) = \bot$. In this case, we observe that $\min_\pi(\varphi \wedge \psi) = \pi(j)$. Furthermore, it must hold that $\overline{\omega}(\varphi) \cap \overline{\omega}(\pi(j)) = \emptyset$.

Since ψ and $\pi(j)$ have shared complements, $(\varphi \wedge \psi)\vee\pi(j)$ cannot be a tautology. Then, by hypothesis and the fact that δ is a choice function, we have

$$\begin{aligned}\emptyset \neq \overline{\omega}(\varphi) \cap \delta_\pi(\varphi \wedge \psi) &= \overline{\omega}(\varphi) \cap \delta((\varphi \wedge \psi) \vee \pi(j))\\ &\subseteq \overline{\omega}(\varphi) \cap \overline{\omega}((\varphi \wedge \psi) \vee \pi(j))\\ &\subseteq \overline{\omega}(\varphi) \cap \overline{\omega}(\pi(j))\end{aligned}$$

This means that $\overline{\omega}(\varphi) \cap \overline{\omega}(\pi(j)) \neq \emptyset$, and we have a contradiction.

Thus we have shown that δ_π indeed satisfies **(C2)**. ☐

Lemma E.11. *Let $\dot{-}$ be a fully rational contraction on a non-finitary theory $\mathcal{K}$, such that $\dot{-}$ remains in the excerpt $\mathbb{E}$. Then there exist uncountably many fully rational contractions on $\mathcal{K}$ that remain in $\mathbb{E}$.*

Proof. The proof proceeds analogously to the proof of Lemma E.7.

We have shown that $\mathcal{K}$ contains an infinite cleaving (Lemma 35). Each permutation of this infinite cleaving induces a distinct (Lemma E.6) fully (Lemmas E.9 and E.10) rational (Lemma 38) contraction. By Observation E.4, each of these contractions remains in $\mathbb{E}$. Since there are uncountably many permutations of an infinite set, the result follows. ☐

Theorem 39. *Let $\mathbb{E}$ accommodate rational contraction, and let $\mathcal{K} \in \mathbb{E}$. The following statements are equivalent:*

1. *The theory $\mathcal{K}$ is non-finitary.*
2. *There exists an* uncomputable *rational contraction function on $\mathcal{K}$ that remains in $\mathbb{E}$.*
3. *There exists an* uncomputable *fully rational contraction function on $\mathcal{K}$ that remains in $\mathbb{E}$.*

Proof. We assume we can decide equivalence of formulae in the logic.

- (1) to (2): follows from Lemma E.7.
- (2) to (3): follows from Proposition 29 and Lemma E.11.
- (3) to (2) and (3) to (1): We show these by contraposition. Fix a theory $\mathcal{K}$ with finitely many equivalence classes, with representatives $\alpha_1, \ldots, \alpha_n$. Then one can define a large class of (possibly non-rational) contractions as follows: given φ, decide if φ is equivalent to any non-tautological α_i; if not, return some code word w with $f(w) = \mathcal{K}$; otherwise select some subset X of $\{\alpha_1, \ldots, \alpha_n\}$ such that $Cn(X)$ is in $\mathbb{E}$, and return a code word w with $f(w) = Cn(X)$. The returned code word w depends only on the representative α_i equivalent to φ, not on the syntax of φ. All of these functions are computable, and they include all (fully) rational contractions, so all (fully) rational contractions are computable.

☐

F. Proofs for Section 8 (Effective Contraction in the Büchi Excerpt)

In Section 8, we define a new selection mechanism for contractions that remain in the Büchi excerpt:

Definition 40 (Büchi Choice Functions)**.** *A* Büchi choice function *γ maps each LTL formula to a single Büchi automaton, such that for all LTL formulae φ and ψ,*

(BF1) *the language accepted by $\gamma(\varphi)$ is non-empty;*

(BF2) *$\gamma(\varphi)$ supports $\neg\varphi$, if φ is not a tautology; and*

(BF3) *$\gamma(\varphi)$ and $\gamma(\psi)$ accept the same language, if $\varphi \equiv \psi$.*

In order to show that this selection mechanism gives rise to rational contractions, we connect it to the choice functions (Definition 9) underlying exhaustive contraction functions (Definition 10). Recall from Lemma 20 that a certain kind of traces, the ultimately periodic traces ($\pi \in UP$), correspond to complete consistent theories $Th_{UP}(\pi)$ of LTL. A Büchi choice function γ thus induces a choice function that selects all CCTs corresponding to ultimately periodic traces in the chosen Büchi automaton's accepted language:

Definition F.1. *The choice function δ_γ induced by a Büchi choice function γ is the function*

$$\delta_\gamma(\varphi) = \{\, Th_{UP}(\pi) \mid \pi \in UP \cap \mathcal{L}(\gamma(\varphi)) \,\}$$

Lemma F.2. *If γ is a Büchi choice function, then δ_γ is a choice function, i.e., satisfies **(CF1)** - **(CF3)**.*

Proof. Let γ be a Büchi choice function, which by definition satisfies **(BF1)** - **(BF3)**. We show each of the required properties for δ_γ.

(CF1): By **(BF1)**, the language of $\gamma(\varphi)$ is non-empty. Per a classical result, any Büchi automaton $\gamma(\varphi)$ that recognizes a non-empty language must accept at least one ultimately periodic trace π. Hence, we have $Th_{UP}(\pi) \in \delta_\gamma(\varphi)$, and $\delta_\gamma(\varphi)$ is non-empty.

(CF2): Let φ be non-tautological, i.e., $\varphi \notin Cn(\emptyset)$. By **(BF2)**, it follows that $\gamma(\varphi)$ supports $\neg\varphi$. This means that every trace $\pi \in \mathcal{L}(\gamma(\varphi))$ satisfies $\neg\varphi$. In particular, this holds for every ultimately periodic $\pi \in \mathcal{L}(\gamma(\varphi))$. For such a π, it then follows that $(\neg\varphi) \in Th_{UP}(\pi)$, or equivalently, $\varphi \notin Th_{UP}(\pi)$, and $Th_{UP}(\pi) \in \overline{\omega}(\varphi)$. We have thus shown that $\delta_\gamma(\varphi) \subseteq \overline{\omega}(\varphi)$ holds.

(CF3): Let $\varphi \equiv \psi$. By **(BF3)**, it follows that $\mathcal{L}(\gamma(\varphi)) = \mathcal{L}(\gamma(\psi))$ holds. It is then easy to see, from the definition of δ_γ, that also $\delta_\gamma(\varphi) = \delta_\gamma(\psi)$ holds.

We conclude that δ_γ is indeed a choice function. ☐

Corollary F.3. *The Büchi contraction function induced by γ is an exhaustive contraction function, with the underlying choice function δ_γ.*

Proof. This follows directly from Lemma F.2, and rewriting $\mathcal{S}(\gamma(\varphi))$ as $\bigcap \delta_\gamma(\varphi)$, using Lemma 21. ☐

To see that Büchi contraction functions remain in the Büchi excerpt, we show the following general property for the support of Büchi automata.

Lemma F.4. *Let A_1, A_2 be Büchi automata. Then it holds that $\mathcal{S}(A_1) \cap \mathcal{S}(A_2) = \mathcal{S}(A_1 \sqcup A_2)$.*

Proof. Let $\varphi \in \mathcal{S}(A_1) \cap \mathcal{S}(A_2)$. By definition of support, this means that $\pi_1 \models \varphi$ for each $\pi_1 \in \mathcal{L}(A_1)$, and $\pi_2 \models \varphi$ for each $\pi_2 \in \mathcal{L}(A_2)$. It is easy to see that this is equivalent to the statement that $\pi \models \varphi$ for each $\pi \in \mathcal{L}(A_1) \cup \mathcal{L}(A_2)$. As $\mathcal{L}(A_1 \sqcup A_2) = \mathcal{L}(A_1) \cup \mathcal{L}(A_2)$, and by definition of support, the latter statement is in turn equivalent to $\varphi \in \mathcal{S}(A_1 \sqcup A_2)$. We have thus shown the equality. ☐

Corollary F.5. *Let $\dot{-}_\gamma$ be a Büchi contraction function on a theory $\mathcal{K} = \mathcal{S}(A)$, where A is a Büchi automaton. The contraction $\dot{-}_\gamma$ satisfies $\mathcal{K} \dot{-}_\gamma \varphi = \mathcal{S}(A \sqcup \gamma(\varphi))$ if $\varphi \in \mathcal{K}$ and φ is not a tautology, or $\mathcal{K} \dot{-}_\gamma \varphi = \mathcal{S}(A)$ otherwise. Hence, $\dot{-}_\gamma$ remains in the Büchi excerpt.*

Proof. This follows directly from Definition 41 and Lemma F.4. □

We have thus shown that BCFs are rational and remain in the Büchi excerpt. Let us now consider the opposite direction. We make use of the fact that every rational contraction is an ECF induced by some choice function δ.

Definition F.6. *Let $\mathcal{K} = \mathcal{S}(A)$, for a Büchi automaton A, and let $\dot{-}_\delta$ be a rational contraction on $\mathcal{K}$ that remains in the Büchi excerpt, induced by a choice function δ. We define the Büchi choice function γ_δ, such that for each formula φ:*

$$\gamma_\delta(\varphi) = \begin{cases} A_\varphi & \textbf{\textit{if}}\ \varphi \in Cn(\emptyset) \\ A_{\neg\varphi} & \textbf{\textit{if}}\ \varphi \notin \mathcal{K} \\ A' & \textbf{\textit{else}}\textit{, where } \mathcal{S}(A') = \bigcap \delta(\varphi) \end{cases}$$

Lemma F.7. *Let $\mathcal{K} = \mathcal{S}(A)$, for a Büchi automaton A, and let $\dot{-}_\delta$ be a rational contraction on $\mathcal{K}$ that remains in the Büchi excerpt, induced by a choice function δ. The function γ_δ is a well-defined Büchi choice function.*

Proof. First, we show that an automaton A' as in the definition always exists. Let us thus assume that $\varphi \in \mathcal{K}$ is non-tautological. Since $\dot{-}$ remains in the Büchi excerpt, we know that $\mathcal{K} \dot{-} \varphi = \mathcal{S}(A'')$ for some Büchi automaton A'. We define A' as a Büchi automaton that recognizes precisely the language $\mathcal{L}(A'') \setminus \mathcal{L}(A)$. Such an automaton can always be constructed from A and A''. It follows that A' is unique, up to language-equivalence of automata.

Clearly, we have $\mathcal{L}(A'') = \mathcal{L}(A) \cup \mathcal{L}(A')$. By Lemma 21, this implies $\mathcal{S}(A) \cap \mathcal{S}(A') = \mathcal{S}(A'') = \mathcal{K} \dot{-} \varphi = \mathcal{K} \cap \bigcap \delta(\varphi)$. Since the support depends only on the language of an automaton, this implies $\mathcal{K} \dot{-} \varphi = \mathcal{S}(A'') = \mathcal{S}(A \sqcup A')$. As $\mathcal{S}(A) = \mathcal{K}$, and as the decomposition of $\mathcal{K}$ is necessarily disjoint from $\delta(\varphi)$, it follows that $\mathcal{S}(A') = \bigcap \delta(\varphi)$.

It remains to examine the conditions **(BF1)** - **(BF3)**.

(BF1): In the first two cases, it is again easy to see that $\gamma_\delta(\varphi)$ accepts a non-empty language. In the third case, since $\dot{-}$ is rational, we have $\delta(\varphi) \neq \emptyset$, and consequently the language of A' must be non-empty.

(BF2): We assume φ is non-tautological, so the first case is ruled out. In the second case, clearly $A_{\neg\varphi}$ supports $\neg\varphi$. In the third case, we know that δ satisfies **(CF2)**, i.e., $\delta(\varphi) \subseteq \overline{\omega}(\varphi)$. Thus we have $(\neg\varphi) \in \bigcap \overline{\omega}(\varphi) \supseteq \bigcap \delta(\varphi) = \mathcal{S}(A')$.

(BF3): This follows directly from the definition of γ_δ and the fact that δ satisfies **(CF3)**.

Thus we have shown that γ_δ is a Büchi choice function. □

Theorem 42. *A contraction function $\dot{-}$ on a theory $\mathcal{K} \in \mathbb{E}_{\text{Büchi}}$ is rational and remains within the Büchi excerpt if and only if $\dot{-}$ is a BCF.*

Proof. We have already shown that BCFs are rational (by Corollary F.3 and Theorem 11) and remain in the Büchi excerpt (Corollary F.5).

For the opposite direction, let $\dot{-}$ be a rational contraction on $\mathcal{K}$ that remains in the Büchi excerpt. By Theorem 11, $\dot{-}$ must be induced by a choice function δ. We have shown in Lemma F.7 that the corresponding function γ_δ is a Büchi choice function. From the definition of BCFs and ECFs, it is easy to see that $\dot{-} = \dot{-}_{\gamma_\delta}$. □

Theorem 44. *Let $\dot{-}$ be a rational contraction function on a theory $\mathcal{K} \in \mathbb{E}_{\text{Büchi}}$, such that $\dot{-}$ remains in the Büchi excerpt. The operation $\dot{-}$ is computable iff $\dot{-} = \dot{-}_\gamma$ for some computable Büchi choice function γ .*

Proof. Let $\mathcal{K} = \mathcal{S}(A)$ for a Büchi automaton A, and let γ be a computable Büchi choice function. Recall that it is decidable whether a given φ is tautological (by deciding equivalence with $\top$), and whether $\varphi \in \mathcal{K}$ (Theorem 24). Further, if φ is neither tautological nor absent from $\mathcal{K}$, we have $\mathcal{K} \dot{-} \varphi = \mathcal{S}(A \sqcup \gamma(\varphi))$, and there exists an effective construction for the union operator $\sqcup$. It is then easy to see from Definition 41 that the BCF $\dot{-}_\gamma$ is computable.

To see that computability of the Büchi choice function is necessary, suppose a given contraction $\dot{-}$ is computable, rational, and remains in the Büchi excerpt. By rationality, $\dot{-}$ is an ECF induced by some choice function δ. Then it is easy to see that $\dot{-} = \dot{-}_{\gamma_\delta}$, for the Büchi choice function γ_δ. And in fact, this Büchi choice function γ_δ can be computed: We can decide which case is applicable (again, by decidability of tautologies and membership in $\mathcal{K}$), and the respective automata can be constructed. Of particular interest, in the third case, we can construct the automaton A' as the difference of the automaton A'' supporting $\mathcal{K} \dot{-} \varphi$ (which is computable) and the automaton A (cf. the proof of Lemma F.7). □